MULTIPLYING AND DIVIDING FAST

ESSENTIAL TACTICS TO SPEED UP
MENTAL CALCULATIONS WITH WHOLE NUMBERS

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MULTIPLYING AND DIVIDING FAST. ESSENTIAL TACTICS TO SPEED UP MENTAL CALCULATIONS WITH WHOLE NUMBERS

This book is dedicated with infinite gratitude to my mother, an exemplary educator and human being, and to the inspiring teachers and professors who instilled in me the love of learning and the passion to teach. I have them to thank for the many ways in which I have improved over the years as a communicator, researcher, and instructor. Their imprint lives in every page of this book.
Of any mistakes or inaccuracies in content, method, or strategy, they are, naturally, completely innocent.

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PREFACE

This book will not teach you how to multiply or divide.

I assume that you know how to do that already…. with pen and paper, given ample time.

The goal of this book is to help you improve the speed at which you multiply and divide, without pencil and paper.

Learning to multiply and divide on your feet, fast, will save you time, give your brain a workout, and maybe help you impress others. You'll benefit from the teachings of this book not just the next time you are doing math homework, but also anytime you are splitting a bill, or figuring out final prices of multiple items.

Doing math calculations fast is comparable to reading or typing. Just as you would save enormous amount of time, be able to accomplish more with less effort and in shorter time reading or typing fast, in every area of your life, so can you be more effective, efficient and well regarded if you can do fast mental calculations.

The applications of the most essential of arithmetic operations are innumerable.

Mental arithmetic is not just useful in middle and or high school where the applications may be the most obvious. Even after you've graduated, or if you didn't graduate and you are in the workforce, whether you are going to college or graduate school, whichever your major, fast mental arithmetic can save you precious next time you are taking that time-sensitive standardized test.

You may find it incredibly useful and time saving advanced college math courses like linear algebra where you still need to do arithmetic to do matrix operations for example. Business, science, social science major? You'll probably use multiplying and dividing quite a bit. Even in fields not traditionally thought of as math related, mathematics is becoming increasingly important.

Beyond school, whether you are a contractor, or designer, an educator or business leader, whether you are a lawyer or an accountant, whatevere the trade or profession, you will benefit from these techniques immensely. But even beyond work, mathematics is present in most anyones' everyday activities.

If you cook, workout, exercise, travel, purchase or sell goods and services you are using math.

Can you estimate proportions using multiplication to determine exactly how much of each ingredient should go into a cake? Or the quantities of each ingredient that should be bought at the supermarket, or for that matter, how much money they will cost, or even, how much money you should budget in the first place before you head out to the store.

Can you quickly estimate how much you should be getting after all those hours you worked, or quickly size up cost of living rough estimates when comparing multiple locations to move to? Or if you are estimating monthly expenses, can you quickly provide an estimate of what you are likely to spend during a year, or in evaluating a job offer, can you quickly think of how much money you'll be getting monthly? Or if you are estimating time, how much time will it take you to complete a task or arrive at a certain place.

I'm belaboring the point, which is: math is everywhere, and you do – or should – use it every day to optimize your time and do things better, faster, and more effectively.

Adding and subtracting fast is the essential first step to improving your mental speed and accuracy in arithmetic, if you want to practice adding and subtracting fast, I strongly recommend you get that you read the first book in this series covering fast mental arithmetic techniques for subtraction and additon. But if you are comfortable enough with fast mental addition and subtraction, and you want to take your arithmetic skills to the next level, this may as well be a great starting point.

Whatever your reason for coming on this journey with me, know that I fully believe it's worth the time and effort.

Welcome aboard.

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INTRODUCTION

This book is an effort to combine the various scattered mental arithmetic techniques and strategies that have been taught by teachers in different and distant corners of the world, at different points in time, into a consistent and comprehensive system that will enable you to calculate two, three, and four-digit products and quotients, with stunning speed and without writing anything down.

Fast mental arithmetic, requires, as does most any other skill, consistent practice, and a little flexibility. Math, like languages, and many other arts, cannot be mastered without investing significant time doing it.

Multiplication and its inverse operation, division, are no exception to this rule. The approach we have adopted in this book is that of including both theoretical chapters to present the information, and practice chapters – which are essential to achieve mastery, efficiency in memory and speed, as well as an appendix, to see worked through examples so that you can see one way of thinking through the calculations.

Theoretical chapters are used to present the techniques, explaining the logic behind them, along with one or more examples, while practice chapters are intended to engage you in applying those skills.

The appendix with worked out answers is intended to show you, step by step how the system outlined in the book is applied to the actual examples we ask you to solve in practice chapters. The appendix should be read in tandem with the practice chapters since the exact steps we take in the appendix are outlined there. You should of course conscientiously try to work exercises out on your own, but it will be useful, after you've done that, that you go to the appendix to compare with our proposed strategy so that you may evaluate which is best in the end.

Its worth noting, that reading this whole book and doing all the practice exercises several times is no guarantee you will come out multiplying and dividing incredibly fast.

Beyond the space limitation, success will depend on many other things such as your previous background and the actual attention you are paying to what you are doing (the speed and concentration required for reading a mathematics methods book, even one as simple as this one, is different from that required of a fiction novel or a social science textbook). How much you get out of this book, will be function in great measure, of what you put into it, and how seriously and consistently you apply these techniques to your everyday life.

Think of the schema presented here as a system that you can personalize and tweak according to your own previous math background, as you go about practicing on your own in everyday situations. The key to mastery, is practice.

Although I show one, and occasionally two ways of going about an operation, as I say perhaps too many times throughout the book, which actual combination of techniques and sequencing of techniques is optimal for improving your mental calculation speed, will largely depend on the background with which you come to this book, as well as what techniques you enjoy most.

For most people multiplying 2 by 2 is easier than multiplying 4 by 3, or for that matter, for some people multiplying 3 by 4 is easier than multiplying 4 by 3 - but this is not always the case... Use the material presented here as it best fits your personal skill using your judgement. Do not be a passive observer. Engage.

I would also like to stress before you begin, that it is imperative that you study by heart single digit products, that is the tables of 1 through 9, and as introduced in the first chapter preferably through the table of 20. I cannot stress how much time it will save you, especially when dealing with three, four or more-digit products and quotients. Knowing the basic operations like the back of your hand is essential if you are ever going to achieve mastery in mental products and quotients.

Except for the illustration of mental aligning numbers in the theoretical chapters, all number products and quotients, are presented horizontally. This is done not to save space, but to illustrate the sequential thinking you should be doing as you are calculating to gain speed and accuracy. You see the idea here, is that you think through a math operation in a very linear way, with very little overhead in the form of memorization, so as not to interrupt your mental computing, minimize errors, and optimize time.

Very seldom will you be required to retain digits in memory, but with operations with several digits and/or terms, this may be necessary sometimes. The approach however, is one of streamlining thought, and resorting to memory, only whenever necessary.

Also, you will notice that as we move to more complicated calculations, we will start to assume that you have assimilated some of the most basic tactics and concepts, and we will not spell them out thoroughly. By the time you are comfortable doing three-digit products, we will not stop to review how useful knowing the concept of doubling may be in calculating 3x16, or how a given technique is applied to multiplication and extended to division.

I do include clues sometimes, to make sure you can work through the exercise, but I try to balance thoroughness with keeping the exercises as fluid as possible, assuming you've done the work, have become proficient in the techniques presented and are ready to move forward.

Finally, although I have given the techniques names, these are personal, arbitrary, and functional-intuitive – that is, the names are based on what I thought the technique should be called based on what we are doing with the numbers. Many of these techniques may have received different names in different times, in different parts of the world by different instructors. The names should help you identify the method and logic to apply, but after you have the techniques down, the idea is that you don't have to say in your head as you are doing the math…. "now I'm going to 'round and compensate'", or even less so "now I'm going to 'balance' these numbers since they are multiples of one another to simplify the calculation".

Instead, get familiar with the techniques and apply them, and think as you are doing the mental calculations, what the relationships between numbers is. Think of numbers as abstractions, ideas. Eventually they should just be a fleeting image in your mind as you do the math.

Are you ready?

Let´s begin.

CHAPTER 1: FUNDAMENTAL TECHNIQUES FOR MULTIPLICATION

We start out by laying the foundations for multiplication by reference to single digit products.

The techniques introduced here will be extremely useful not only with larger figure multiplication, but also with division, since many division techniques involve using multiplication in some form or another.

The chapter begins by covering fundamental general rules in the context of single and some double-digit products.

Read this chapter conscientiously as it is the very foundation for everything else that comes after. Although this should be a simple read, internalizing the concepts and strategies laid out here will be key for success later.

MULTIPLICATION FUNDAMENTALS

Because of the commutative property feel free to rearrange the order of the terms whatever multiplication comes your way in whichever way is easiest to calculate.

Because most of us are more familiar with the *table of 2* or the *table of 3*, than we are with the *table of 7 or 8*, when multiplying single-digit numbers it might be easier to reverse terms if necessary and mentally calculate 3 x 8 instead of 8 x 3.

At the same time, it is imperative also that you become familiar with multiplication tables. Especially products involving terms 1 through 10, and preferably 1 through 20.

While we won't cover single digit by single digit products, review them if you feel weak recalling those products. We'll cover however products from 11 to 20 by 1 to 20. Knowing these as well as knowing single digit products, will give you an immense edge moving forward.

x	2	3	4	5	6	7	8	9	10
11	22	33	44	55	66	77	88	99	110
12	24	36	48	60	72	84	96	108	120
13	26	39	52	65	78	91	104	117	130
14	28	42	56	70	84	98	112	126	140
15	30	45	60	75	90	105	120	135	150
16	32	48	64	80	96	112	128	144	160
17	34	51	68	85	102	119	136	153	170
18	36	54	72	90	108	126	144	162	180
19	38	57	76	95	114	133	152	171	190
20	40	60	80	100	120	140	160	180	200

x	11	12	13	14	15	16	17	18	19	20
11	121	132	143	154	165	176	187	198	209	220
12	132	144	156	168	180	192	204	216	228	240
13	143	156	169	182	195	208	221	234	247	260
14	154	168	182	196	210	224	238	252	266	280
15	165	180	195	210	225	240	255	270	285	300
16	176	192	208	224	240	256	272	288	304	320
17	187	204	221	238	255	272	289	306	323	340
18	198	216	234	252	270	288	306	324	342	360
19	209	228	247	266	285	304	323	342	361	380
20	220	240	260	280	300	320	340	360	380	400

When working with double and triple digit numbers, we may recommend techniques that do not benefit from switching terms to make the multiplication more manageable.

Why? This is because when you deal with a three or four-digit product, you may need to work through products taking figures apart, such that reshuffling the group of digits that make up a term, may make the mental computation more complicated.

That said, feel free to reorder the multiplication terms to do mental calculations when you are doing complex calculations, however is most convenient to you.

MULTIPLING BY TWO AS DOUBLING

The table of two is something you are probably very familiar with. Hopefully you can recite it without much thought. But I want to pause here and encourage you to think of the table of two as more than simply a verse that you know by heart. Multiplying by two is ***doubling the number*** you are multiplying.

2x2 is **4**, which is the same as ***doubling the number*** 2.

8x2 is **16**, so is 8+8, or 8 ***doubled***.

If you multiply 16 by 2, it might be easier to think of it as an addition instead of a multiplication.

16x2 might be harder to think of than 16 + 16.

Same if you multiply 173 by 2. Think of 173 doubled instead. If already can add and subtract fast, then this should be no problem. By the way, 173 + 173 is **346**.

Example 1	16x2 = 32

Example 2	242x2 = 484

Example 3	354x2 = 708

DOUBLING TWICE OR THREE TIMES.

Similarly, you can think of multiplying any number by 4, say 16x4 as simply ***doubling twice***. Why? Because any number by 4 is really any number x 2, twice.

So instead of calculating 16 x 4 mentally remembering carries, think of 16 doubled, which is 32, doubled, **64**.

We could extend this indefinitely, depending on your background and needs. But multiplying 16x8, might be more daunting than thinking. 16 doubled is 32, double to 64, double to **128**.

16 x 16 is 16 doubled, 32, doubled (x4), 64, doubled (x8) 128, doubled (x16) **256**.

That's much easier than doing it with the traditional algorithm.

16 x 16 = 096 + 160 = **256**… especially since there are carries.

You can use the doubling technique when multiplying by 2, 4, 8, 16, 32, 64, 128, 256, 312, 624 or any other powers of two. Note that 4 is 2^2, 8 is 2^3, 16, 2^4, and so on and so forth.

Example 1	5x8 10, 20, 40 40

Example 2	42x16 84, 168, 336, 672 672

Example 3	304x4 608, 1,216 1,216

MULTIPLES OF 10

The ***multiples of 10 rule*** is simple enough. 8 x 8 is **64**. What about 8 x 80. It's **640**. You just append to the result however many zeros the multiple of 10 multiplicator has, like so:

3 x 40 is 3x4, and a trailing zero, **120**.

9 x 80 is 9x8, and a trailing zero, **720**.

So, 16 x 20 is only 16 x 2, or 16 doubled, which is 32, adding a zero in the ones place value, **320.**

Its indistinct whether the multiple of ten is the multiplicator or multiplicand. That is the first or second term in the multiplication.

By the *commutative law* 20 x 16 is the same as 16 x 20.

Also, 30 x 50 is **1500**. Really, it's just 3 x 5 = 15, plus a zero for each of the place values that have a zero in this case two: the ones place in 30 and the ones place in 50.

What about 16,000 x 2,000, its 32 plus 6 trailing zeros. So, **32 million** (or $\mathbf{32x10^6}$)

Integrating this technique with the doubling technique we learned we know that 16,000 x 16,000, is really 16 doubled, 32, doubled, 64, doubled, 128, doubled, 256 and six trailing zeros, **256 million**.

Example 1	7 x 410 = 2,870

Example 2	50 x 90 = 4,500

Example 3	40 x 45,000 = 1,800,000 (1.80×10^6)

MULTIPLYING BY 5 (HALVING)

Knowing that anything multiplied by 5 is half of what it would be multiplied by 10 will make things easier further down the road. We'll get to that later in this section, but first, let's notice some things about multiplying by 5 that will be useful later.

Multiplying any whole number by 5 will always result in a number that has 0s or 5s in the ones digit place value.

5x**1** = **5**
5x2 = 10
5x**3** = 1**5**
5x4 = 20
5x**5** = 2**5**

From this pattern we can also notice that when multiplying 5 times an even number, we'll get a 0 in the ones place value, and when multiplying 5 times an odd number, we'll instead get a 5 in the ones place value.

5 x 1**5** = 7**5**
5 x 192 = 960
5 x 37**3** = 1,86**5**
5 x 2,388 = 11,940

And so on, and so forth.

We'll review some implications of this rule later. Remember we are covering foundations.

But for now, just notice that any multiple of 5, ending in 5 (any whole number with 5 in the ones place value) like 15, 25, 135, 2,345 will follow this rule also.

15 x **7** = 10**5**
25 x 16 = 400
135 x 4**1** = 5,53**5**
2,345 x 58 = 136,010

Now, on to the actual technique we'll use.

5 x 48 might seem daunting. But since 5 times anything is half of 10 times that something, then we can just do 10 x 48, which is 48**0**, halve it, 24**0**. We got there a lot quicker!

5 x 3**7** is **10** x 37 / 2, and 37**0** / 2 is **185**
5 x 96 is **10** x 96 / 2, and 96**0** / 2 is **480**
5 x 1**4** is **10** x 14 / 2, and 14**0** / 2 is **70**

Knowing this will be useful when we multiply by 15.

Example 1	5 x 74 = 740 / 2 = 370

Example 2	5 x 810 = 8100 / 2 = 4050 4050

Example 3	93 x 5 = 5 x 93 = 930 / 2 = 465

SQUARES ENDING IN 5

Whenever you have a double-digit number ending in 5 and you multiply it by itself, there's a little trick that will come in very handy.

The tens digit of the multiplicator (or multiplicand, since in this case they are the same) will determine what the leftmost place value digits will be by

n x (n+1)

where n is the tens digit of the multiplicator.

And the tens place value digit of the product will always be two, and the ones place value digit will be five, so, *25 in the ones*, always.

Thus in,

15 x 15

The left most place value digits will be 1 x (1+1) = 1 x 2 = **2**, the rightmost digits will be 25.
So the result will be **2**25

25 x 25

The left most place value digits will be 2 x (2+1) = 3 x 2 = **6**, the rightmost digits will be 25.
So the result will be **6**25.

And in,

195 x 195

The left most place value digits will be 19 x (19 + 1) = 19 x 20 = **380** (note that 19 x 10 is 190, doubled, 380, since 19 x 20 is the same as 19 x 10 x 2), the rightmost digits will be 25.
So the result will be **38,0**25.

And so on and so forth.

Example 1	35 x 35

	3 x 4 = 12 1,225

Example 2	105 x 105 10 x 11 = 110 11,025

Example 3	95 x 95 9 x 10 = 90 9,025

MULTIPLYING BY 11 – TWO DIGITS up TO 90

Multiplying by any one-digit number by 11 might not seem so hard. But what if you have a figure with more digits. Let's look at some examples.

11 x 11 = 121
11 x 23 = 253
11 x 43 = 473
11 x 81 = 891

Do you notice a pattern?

In the previous products, the hundreds place value digit corresponds with the tens place value digit of the multiplicand, and the ones place value digit in the result corresponds with the ones value digit in the multiplicand.

11 x 11 = 121
11 x 23 = 253
11 x 43 = 473
11 x 81 = 891

What about the tens digit place value figure?

It looks like it is the result of the ones place value plus the tens place value digit in the multiplicand.

11 x 11 = 121 (1+1 = 2)
11 x 23 = 253 (2+3 = 5)
11 x 43 = 473 (4+3 = 7)
11 x 81 = 891 (8+1 = 9)

So that's quite useful. Let's multiply something else.

11 x 46 = 506

In this example everything is fine, except instead of a four in the hundreds place value, we have a five,

the reason being 4+6 is 10. So, whenever adding the tens and ones in the multiplicand is greater than 9, we use the ones place value of the addition for the tens place value in the result, and we add whatever units must be added that we carried over to the number that would have gone in the hundreds.

In this case we add 1 that we carried from 4+6 to the 4, resulting in 5 instead in the hundreds place value.

Example 1	11 x 72 = 792

Example 2	11 x 38 = 418

Example 3	11 x 55 = 605

We can apply this rule to any number up to 90. We'll look at what happens with larger figures in the next article.

MULTIPLYING THREE OR MORE DIGIT FIGURES BY 11

The ***rule of multiplying by 11*** can be extended further for results larger than three digits.

Let's look at an example,

11 x 98 = 1,078

Notice the 8 remains in the ones just as previously.
While 9 should have been in the hundreds, because 9+8 is 17, we get a 10 in the hundreds (that is a one in the thousands and a 0 in the hundreds place value) and the seven in the tens.

11 x 98 = 1,078 (9 + 8 = 17)

That's not too hard, but what about this product,

11 x 987 = 10,857

Again, the ones place value corresponds to the last digit in the multiplicator, 7. But something different happens in the middle. Because we have more than two figures we add in pairs.

Like so

11 x **98**7, 9+8 = 17
11 x 9**87**, 8+7 = 15

You can see the 5 occupies the tens place value.
And then the 7 + the 1 from 15, the hundreds.
Finally, the 10 in the result is the result of adding 1 from the tens place value figure in 17 to the leftmost digit in the multiplicand, 9. So mentally aligning, that's

9000 (nine in the thousands)
1700 (seventeen in the hundreds)
0150 (fifteen in the tens)
0007 (seven in the ones)

10,857.

When dealing with multiplication by 11 where the multiplicator is at least three digits, mentally align the numbers following these steps.

1) Add the ones and the tens, note if there are carries. You know now what the ones and the tens in the result will be.

2) Add the tens and the hundreds adding the carries from the previous, note if there are carries. You will now know what the hundreds and thousands place values will be.

Extend this further if you are dealing with four digits (or more) for instance.

11 x 7,246 = 79,706

1) Add the ones and the tens. Note there's 1 carry. You now know that 6 is in the ones, and 0 in the tens, because 4+6 is 10. The one is the carry the 0 stays in the tens.

11 x 7,246 = 79,706
4+6 = **10**

2) Add the tens and the hundreds, 2+4 = 6, plus the one you carried, 7 in the hundreds.

11 x 7,246 = 79,706
2+4+**1** = **7**

3) Add the hundreds and the thousands 7+2 = 9, plus no carries, 9 in the thousands.

11 x 7,246 = 79,706
7+2 = **9**

4) 7 stays as the digit in the leftmost place value, the ten-thousands, because there were no carries in the previous sum.

11 x **7**,24**6** = **7**9,70**6**

If you wanted to align, it would be something like,

70000 (seven in the ten-thousands)
09000 (nine in the thousands)
00600 (six in the hundreds)
00100 (ten in the tens)
00006 (six in the ones)

79,706.

Example 1	11 x 3,480 8 + 0 = 8 4 + 8 = 12 3 + 4 = 7 30000 07000 01200 00080 00000 38280

Example 2	11 x 2,911 1 + 1 = 2 9 + 1 = 10 2 + 9 = 11 20000 11000 01000 00020 00001 32,021

Example 3	11 x 4,257 5 + 7 = 12 2 + 5 = 7 4 + 2 = 6 40000 06000 00700 00120 00007 46,827

MULTIPLYING BY 15 (DECOMPOSING AND HALVING)

Advanced techniques for quick mental multiplication address making large digit figures manageable.

Multiplying by 15 is a case in point.

Given the following problem:

15 x 17

We know that 5 x odd number, will result in a *five in the ones*.

Note also that because of the distributive property, we can rethink 15 x 7 as
10 x 7 + 5 x 7, because
7 x 15 =
7 x (5 + 10) =
7 x 5 + 7 x 10

Then,

10 x 7 = 70 & 5 x 7 = 35

And, 70 + 35 = **105**

We can break down multiplication by 15 for any other case like so.

15 x 21 is really

10 x 21 = 210
5 x 21 = 105

And, 210 + 105 = **315**

Pretty neat.

15 x 37 can be calculated as

10 x 37 = 370
5 x 37 = 185

But if 5 x 37 sounds daunting to do on your mind, note that 10 x 37 / 2 is the same.

So, halve 370 to 185. And get that answer more quickly.

370 + 185 = **555**

Example 1	15 x 88 10 x 88 = 880 880 /2 = 440 880 + 440 = 1,320

Example 2	15 x 27 10 x 27 = 270 270 /2 = 135 270 + 135 = 405

Example 3	15 x 382 10 x 382 = 3820 3820 / 2 = 1910 3820 + 1910 = 5,730

CHAPTER 2: GENERAL TECHNIQUES FOR MULTIPLICATION

In this chapter we'll cover the techniques you must be familiar with to tackle the practice section in Chapter 3. The techniques presented here work their magic for single, double, triple digit products and beyond.

FROM LEFT TO RIGHT – MOSTLY

The traditional algorithm, the method you most likely were taught at school, requires you to multiply from the ones place value and move toward the tens, hundreds and so on.

While this has the advantage of systematizing the procedure and minimizing error it is useful only if is done on paper, since thinking the number backwards (from the ones moving toward the thousands) makes it very difficult for you to retain the numbers and invoke them in reverse order (from the thousands to the ones place value).

For this reason, we recommend moving from left to right.

This is not a fixed rule and there are exceptions. For example, when we looked at the ***rule of multiplying by 11*** for three-digit results.

That said, it is generally easier to remember the result of your calculation moving from left to right.

This is to say, quite simply, that if you need to remember a figure such as 1,375, its best to calculate the figure in the leftmost place value, like, a 1 in the thousands.

In the example just mentioned, 1,375, you would arrive to 1 first, then to the 3 in the hundreds. By the time you arrive to 3, recalling 1 will not be hard (you just calculated it!), and you can even say in your head 1 thousand 3 hundred… then arrive at a 7, seventy, 137…, and at a 5: **1,375**.

No need to reverse the figures in your mind.

This also has the added benefit, that if all you need is to approximate a figure (especially when dealing with millions and beyond), you don't have to calculate the whole figure to know whether you are dealing with 1 or 9 million.

TWO DIGIT FIGURE PRODUCTS

When you are multiplying a *two-digit figure* by another *two-digit figure*, and especially when you are multiplying figures that contain single digits whose individual products do not result in carries, for example 12 x 13, 59 x 11, or 14 x 21, use the following technique.

Example

12 x 24

1. Multiply the tens place value figures for the multiplicator and multiplicand. **1** x **2** = **2**
2. Multiply the ones place value figures of the multiplicator and multiplicand. **2** x **4** = **8**
3. Multiply the multiplicators' ones' place value figure by the multiplicands' tens' place value figure, multiply the multiplicators' tens' place value figure by the multiplicands ones' place value figure, add the two
   1 x **4** + **2** x **2** = **8**
4. Finally align them like so, the result of step one will be the product's leftmost place value figure, the result of Step two will the product's ones' place value figure, the result of step 3 will be the result of the tens place value figure.

200
0**8**0
00**8**

288 is the result

Let's try it again, this time, without instructions, to make it more intuitive to see how quick the process should be in your head.

42 x 12

4 x 1 = **4**
2 x 2 = **4**
4 x 2 + 1 x 2 = **10**

400
100
01**4**

504

One final example,

38 x 95

3 x 9 = **27**
8 x 5 = **40**
3 x 5 + 8 x 9 = **87**

2700
0**87**0
00**40**

3,610

Example 1	59 x 12 5 x 1 = 5 9 x 1 = 18 5 x 2 + 1 x 9 = 19 500 190 018 708

Example 2	34 x 65 3 x 6 = 18 4 x 5 = 20 3 x 5 + 4 x 6 = 39 1800 0390 0020 2,210

Example 3	98 x 47 9 x 4 = 36 8 x 7 = 56 9 x 7 + 8 x 4 = 95 3600 0950 0056 4,606

BALANCING FOR MULTIPLICATION

Let’s note that in the multiplication

12 x 15, both terms 12 and 15 are divisible by 3.

Divide by 3 the smallest of the terms 12 / 3 is **4**.
And multiply by 3 the remaining term 15 x 3 is **45**.

We get that

45 x 4, which is much easier to calculate than 12 x 15 are both **180**.

Example 1	144 x 24 = (144 x 2) / (24 / 2) = 288 x 12 = 288 x 10 + 288 x 2 = 2880 + 576 = 3,456

Example 2	45 x 185 = (185 x 5) / (45 / 5) = 925 x 9 = 8,325

Example 3	77 x 252 = (252 x 7) / (77 / 7) = 1,764 x 11 6 + 4 = 10 7 + 6 = 13 1 + 7 = 8 10000 08000 01300 00100 00004 19,404

ROUNDING AND COMPENSATION

A core technique, perhaps the most important for fast addition and subtraction of virtually any number, ***rounding and compensation*** occupies a central place in fast multiplication and division.

The technique consists of ***rounding a number up*** or ***down***, ideally to a multiple or 10, or a multiple of any other number you find manageable. Typically rounding to a multiple of 10 will suffice.

If the numbers you are working with are large enough and you are comfortable with fast addition and multiplication, you may prefer to round 371 not to 370, but to 400 for example.

If you ***round*** a number, you are essentially borrowing some units that must be "***compensated***" later on, to arrive at the true result of the operation.

Applied to multiplication. The ***compensation*** does not consist of simply adding or subtracting what you borrowed to round to the partial result but accounting also for the factor by which you are multiplying.

For instance,

If you are multiplying 96 x 2, it might be easier to think of this operation as doubling. But doubling 96, is harder than doubling the closest multiple of 10, which is 100. So double 100 instead and remember you borrowed 4 units to go from 96 to 100. Then when you compensate, you will not subtract 4, but 8, since you doubled the number after rounding. So, 200 - 8 is effectively the same as 96 x 2.

Or if the product had been

78 x 3. You would have tripled 80, 3x8 is 24, so 3 x 80 = 240, and since you borrowed 2 to round from 78 to 80, you would return not 2, but 6, since you tripled the result.

240 - 6 = 78 x 3

Once you get the hang of it, it will be much easier and more fun.

If you practiced rounding and compensation from our previous series in addition and subtraction, then the technique here may be easier to apply.

Example 1	7 x 91 (7 x 90) + (1 x7) = 637

Example 2	16 x 191 (16 x 200) - (9x16) 3,200 - 144 = 3,056

Example 3	88 x 17 (90 x 17) - (17 x 2) 1530 – 34 = 1496

ASSOCIATIVE PROPERTY OF MULTIPLICATION

When dealing with products, its useful to know its divisors and multiples.

We know that 56 x 33 is the same as 56 x 11 x 3.

And calculating the latter might be easier than the former.

Practice breaking down numbers into a sub-product to make multiplication easier.

The procedure is as follows, first pick *the smallest* of the two numbers and *break it down* into a sub-product.

Try to break the number down into a product that has at least as one of its terms 2, 3, or 5, 7 or any of the lowest possible prime numbers.

Know at least what the first few additive primes are:
2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

For instance,

107 x 84

The lowest number 84, can be broken down (step-by-step here) as follows.

84 = 2 x 42 = 2 x 2 x 21 = 2 x 2 x 3 x 7

Consider the following problem,

1,937 x 74

The smallest number is 74, which can be broken down as follows,

74 = 2 x 37

And here’s why its useful to know two-digit primes.

Here 37 is prime, so it’s only divisible by 1 and itself, and cannot be further broken down as the product of two whole numbers.

Knowing the primes by heart, allows you to see when you cannot further simplify the product.

Let's work through an example from beginning to end.

98 x 56

56 = 2 x 28 = 2 x 2 x 14 = 2 x 2 x 7 x 2

Now that you know this, you can set this up as

98 x 2 x 2 x 2 x 7

Now double 98 three times, applying ***rounding and compensation***.

That's
98 is 100 rounded, 200 doubled, -4 compensated, **196**
196 is 200 rounded, 400 doubled, -8 compensated, **392**
392 is 400 rounded, 800 doubled, -16 compensated, **784**

Again, note here although we rounded 98 to 100, by two units, we compensated by subtracting 4 to reach **196**. The compensation portion of the technique requires that the compensation be proportional to the rounding: if you rounded by adding 4 units, and then doubled, you would compensate by 8 units, not 4.

So

(98 x 2 x 2 x 2) x 7

784 x 7 is **5,488**

For now, we are concerned with products of positive numbers. In this case, when you are multiplying,

if you are rounding up a positive number, you must ***compensate by subtracting***,
if you are rounding down a positive number, you must ***compensate by adding***.

Example 1	48 x 13 = 2 x 2 x 2 x 2 x 3 x 13 13 doubled 4 times: 13 doubled 26, 26 doubled 52 52 doubled 104 104 doubled 208 208 x 3 = 624

Example 2	72 x 35 = 72 x 5 x 7 72 x 5 = 360 360 x 7 = 2,520

Example 3	22 x 267 = 11 x 2 x 267 267 doubled, 534 534 x 11 3 + 4 = 7 5 + 3 = 8 5,874

DISTRIBUTING A TERM FOR DIFFICULT PRODUCTS

The technique of ***Distributing a Term*** is a more general application of the tactic of ***Multiplying by 15***, whereby we multiplied the term by 10 and added that to multiplying the term by five such that for example,

37 x 15 = 37 x 10 + 37 x 5

Extending this further you could apply it to an operation such as

47 x 12 = 47 x 10 + 47 x 2
85 x 13 = 85 x 10 + 85 x 3

And so on, and so forth.

If you are comfortable adding large numbers and multiplying by a single digit number, then you can tackle all sorts of seemingly complex products.

Solving any number multiplied by any other number from 12 to 20 (and beyond if you are so inclined) becomes much easier.

If you are multiplying any number by a two-digit figure from 20 to 99, you can break down the operation by multiplying by ten, which is simple enough, and then by a single digit corresponding to the tens place, and finally adding that to multiplying the term by the ones place, like so,

394 x 38

Can be expressed as

394 x 10 x 3 + 394 x 8

394 x 10 = 3940
3,940 ***round*** to 4,000, 4,000 x 3 = 12,000, ***compensate*** - 180 (which comes from 3 x 60), is 11,820
394 x 8 ***round*** to 400 x 8 = 3200, ***compensate*** -48 (which comes from 8x6), is 3,152
11,820 + 3,152 = **14,972**

471 x 47

can be expressed as

471 x 10 x 4 + 471 x 7

471 x 10 = 4,710
4,710 ***round*** to 5,000, 5,000 x 4 = 20,000, ***compensate*** -1160 (which comes from 4 x 290), is 18,840
471 x 7 ***round*** to 400 x 7 = 2,800, ***compensate*** + 497 (which comes from 7 x 71), is 3,297
18,840 + 3,297 = **22,137**

Example 1	421 x 32 = (400 + 21) x 32 = 400 x 32+ 21 x 32 = 12,800 + 21 x 30 + 21 x 2 = 12,800 + 630 + 42 = 12,800 + 672 = 13,472

Example 2	748 x 77 = (700 + 48) x 77 = 700 x 77 + 48 x 77 = 700 x 70 + 700 x 7 + 48 x 77 = 49,000 + 4,900 + 48 x 77 = 53,900 + 48 x 70 + 48 x 7 = 53,900 + 3,360 + 336 = 53,900 + 3,696 = 57,596

Example 3	51 x 295 = 295 x 10 x 5 + 295 = 2,950 x 5 + 295 = 29,500 /2 + 295 = 14,750 + 295 = 15,045

MULTIPLYING ANY THREE DIGIT FIGURES.

We saw a method for multiplying any two-digit figures. Let’s revisit the example briefly.

42 x 12

4x1 = **4**
2x2 = **4**
4x2 + 1x2 = **10**

400
100
00**4**

504

With multiple carries it is a bit more complicated, but not a lot more so once you gain practice.

But what if we are multiplying 3-digit figures, is there an analog method.

It turns out there is.

Let’s look at an example

123 x 432

The steps are as follows:

1. Hundreds by hundreds = 4 x 1 = 4
2. Add hundreds by tens (1 x 3) + (4 x 2) = **11**
3. Add hundreds by ones and tens by tens (2 x 1) + (2 x 3) + (3 x 4) = **20**
4. Add tens by ones (3 x 3) + (2 x 2) = **13**
5. Ones by ones 3 x 2 = **6**

40000
11000
0**20**00
00**13**0
0000**6**

53,136

Example 1	235 x 278
40000 **20**000 0**47**00 00**59**0 000**40** **65,330**	2 x 2 = 4 2 x 7 + 2 x 3 = 20 2 x 8 + 3 x 7 + 2 x 5 = 47 3 x 8 + 7 x 5 = 59 5 x 8 = 40

Example 2	400 x 481
160000 0**32**000 000**4**00 0000**0**0 00000**0** **192,400**	4 x 4 = 16 4 x 8 + 0 x 2 = 32 4 x 1 + 0 x 8 + 0 x 1 = 4 0 x 1 + 8 x 0 = 0 0 x 1 = 0

Example 3	591 x 696
300000 **100**000 0**107**00 000**63**0 00000**6** **411,336**	5 x 6 = 30 5 x 9 + 6 x 9 = 100 5 x 6 + 9 x 9 + 1 x 6 = 107 9 x 6 + 1 x 9 = 63 6 x 1 = 6

VARIATION: TWO DIGIT FIGURE BY THREE DIGIT FIGURE.

Here the technique remains the same. The only modification is adding zeros to the place values so as to make the algorithm work. Like so,

347 x 38

Treat is as
347 x 038 **(treat the empty space as a zero)**

1. 3 x 0 = 0
2. 3 x 3 + 4 x 0 = 9
3. 3 x 8 + 4 x 3 + 7 x 0 = 36
4. 4 x 8 + 3 x 7 = 53
5. 7 x 8 = 56

00000
09000
03600
00530
00056

13,186

While this technique works, it may be time consuming, so we recommend you stick to a combination of ***rounding and compensation***, the ***associative*** and ***distributive*** strategies to tackle these problems.

Example 1	17 x 948
00000	0 x 9 = 0
0**9**000	0 x 4 + 9 x 1 = 9
0**67**00	0 x 8 + 1 x 4 + 7 x 9 = 67
00**36**0	4 x 7 + 8 x 1 = 36
000**56**	7 x 8 = 56
16,116	

Example 2	628 x 81
000000	6 x 0 = 0
0**48**000	6 x 8 + 0 x 6 = 48
00**22**00	6 x 1 + 2 x 8 + 0 x 1 = 22
000**66**0	2 x 1 + 8 x 8 = 66
00000**8**	8 x 1 = 8
50,868	

Example 3	44 x 452
000000	0 x 4 = 0
0**16**000	0 x 5 + 4 x 4 = 16
00**36**00	0 x 2 + 4 x 5 + 4 x 4 = 36
000**28**0	2 x 4 + 4 x 5 = 28
00000**8**	4 x 2 = 8
19,888	

LARGE PRODUCTS WITH ONE AWAY FROM A POWER 10 MULTIPLIER

When you have a large product where either the multiplicand or multiplicator is one away to a power of 10, especially a large multiple of 10, like 100, 1,000, 10,000 and so on and so forth. There's a general principle you can apply that will save you a significant amount of time.

Let's look at an example

387 x 999

You can skip the traditional algorithm if you observe the following.

The left most place value digits of the product will be the multiplier (or whichever number in the product is not a number close to a power of 10), in this case 387, minus however many units the other figure is missing to reach its closest power of 10. In this case 999 is off 1,000 by 1. So, the leftmost place value digits will be 387 - 1 **= 386.**

The right most place value digits of the product will be whatever closest power of 10 figure we are working with, in this case 1,000, since 999 is closest to 1,000, minus the multiplier (or whichever number in the product is not a number close to a power of 10), in this case 387, times however many digits the figure closest to the power of 10 is. So, 1,000 - 387, is **613.**

So, the product is **386,613**.

Example 1	428 x 999 428 - 1 = 427 (1,000 – 428) x 1 = 572 427,572

Example 2	688 x 998 688 - 2 = 686 (1,000 - 688) x 2 = 312 x 2 = 624 686,624

Example 3	215 x 998 215 - 2 = 213 (1,000 - 215) x 2 = 785 x 2 = 1,570 **213**000 00**1570** 214,570

MULTIPLIER AND MULTIPLICAND CLOSE TO A POWER OF 10

But what if you are dealing with products where both multiplier and multiplicand are close to a power of 10, but perhaps by more than 1, as in 994 x 997 or 91 x 98, or 9,996 x 999

A quick Vedic Math algorithm to find the answer to these products is the following:

1. Find what power of 10 is closest to the multiplier and multiplicator
2. Subtract that number from both multiplier and multiplicator, separately
3. Add the multiplicator to the difference in the multiplicand to reach the power of 10 in question, or since the result will be the same, add the multiplicand to the difference in the multiplicator to reach the power of 10 in question.
4. Finally multiply the differences you took in step 2 by each other.
5. Align the numbers in step 4 to the left, they will be the leftmost values of the product. Note that there should be as many place value digits as there are 0s in the power of 10 in question.
6. Align the numbers in step 3 to the right.
7. Append the figure you obtained in 5 to 6, and align where necessary.

Let's look at three examples, which will make it apparent how simple this is:

994 x 997**,**

Here 1,000 is obviously the power in question, 994 - 1,000 = **-6**, 997 - 1,000 = -3.
Ok, 994 + (-3) is **991**, this then is our leftmost place value figure.

Finally, -6 x (-3) is **18**, but since we are dealing with three zeroes in 1,000 there should be 3 place values.
So, the right most place value figures will be **018**
Join and you have **991,018**, the result

Another example,

91 x 98 =
91 - 100 = -9, 98 - 100 = -2
91 + (-2) = 89
-9 x (-2) = **18**
Result is **8,918**

Remember this works if both multiplier and multiplicand have the same number of digits.

Of course, in this example you could have also used rounding and compensation, like so,

91 x 100 = 9,100
9,100 - 2 x 91 =
9,100 - 182 =

8,918

And maybe, depending on your background and previous knowledge, have arrived at the result faster.

Example 1	96 x 99 96 - 100 = -4, 99 - 100 = -1 96 + (-1) = 95 -4 x (-1) = 4 9500 0004 9,504

Example 2	997 x 998 997 - 1,000 = -3, 998 - 1000 = -2 997 + (-2) = 995 -3 x (-2) = 6 995000 000006 995,006

Example 3	9,995 x 9,997 9,995 - 10,000 = -5, 9,997 - 10,000 = -3 9,995 + (-3) = 9,992 -5 x (-3) = 15 99920000 00000015 99,920,015 (or ≈ 99.91 x 10^6)

CHAPTER 3: PROGRESSIVE PRACTISE I

This section consists of examples. To the right of each exercise I've added references to a possible way to solve it. On the worked out answers in the appendix, which I recommend you read in tandem with this chapter as you solve the exercises, you can see which techniques and essential concepts on every step of the way.

Remember two things going forward.

First, if you have trouble recognizing which basic concept or technique is best suited for each step of the calculation, you may need a better understanding of concepts and techniques. Concepts like ***rounding and compensation***, are naturally more complex because they involve more than one step.

Second, the technique that I've shown may not be the one that comes easiest to you. You may even find you are solving the exercieses in creative ways that combine various techniques that I have not anticipated. As I've said before, which method is best will depend on your familiarity with arithmetic, personal number associations, and several other factors.

PRODUCTS TO MEMORIZE

7x6	(invert)
5x9	(multiples of 10)(halve)
8x7	(double three times)
9x8	(double three times)
9x6	(memorize)
9x7	(memorize)
11 x 17	(memorize)(rule of 11)
12 x 13	(memorize)
13 x 14	(memorize)(decompose)- (multiples of 10) (double twice)
14 x 15	(memorize)(decompose)
15 x 16	(memorize)(decompose)
16 x 17	(memorize)
17 x 18	(memorize)(decompose) - (multiples of 10) (double three times)
18 x 19	(memorize)
11 x 14	(memorize) (rule of 11) (decompose) (multiples of 10) (double twice)
12 x 19	(memorize)(decompose) (multiples of 10)
13 x 18	(memorize)(decompose) (multiples of 10) (double three times)
14 x 17	(memorize) (decompose)- (multiples of 10) (double twice)

DOUBLING AND MULTIPLES OF 10

243 x 2	(double)	79 x 8	(double three times)
384 x 2	(double)	19 x 80	(multiples of 10)
562 x 2	(double)	7 x 400	(multiples of 10)
719 x 2	(double)	7 x 340	(multiples of 10)
17 x 4	(double twice)	3 x 200	(multiples of 10)
38 x 8	(double three times)	9 x 150	(multiples of 10)
139 x 4	(double twice)	5 x 41	(multiples of 10) (halve)
385 x 8	(double three times)	5 x 180	(multiples of 10) (halve)
479 x 4	(double twice)	5 x 77	(multiples of 10) (halve)
13 x 8	(double three times)	5 x 780	(multiples of 10) (halve)
79 x 4	(double twice)	5 x 950	(multiples of 10) (halve)

SQUARES ENDING IN FIVE, AND PRODUCTS WITH ELEVEN

35 x 35	(squares of numbers ending in 5)	11 x 8	(memorize, or rule of 11, or double three times)
15 x 15	(squares of numbers ending in 5)	11 x 313	(rule of 11)
155 x 155	(squares of numbers ending in 5)	11 x 54	(rule of 11)
85 x 85	(squares of numbers ending in 5)	11 x 733	(rule of 11)
75 x 75	(squares of numbers ending in 5)	11 x 89	(rule of 11)
		11 x 220	(rule of 11)
		11 x 35	(rule of 11)

RULE OF FIFTEEN AND TWO DIGIT FIGURE PRODUCTS

15 x 83	(rule of 15)	34 x 52	(Two-digit by two-digit)
15 x 120	(rule of 15)	31 x 42	(Two-digit by two-digit)
15 x 165	(rule of 15)	67 x 81	(Two-digit by two-digit)
15 x 932	(rule of 15)	15 x 35	(Two-digit by two-digit)
15 x 731	(rule of 15)	57 x 89	(Two-digit by two-digit)
94 x 14	(Two-digit by two-digit)		

BALANCING & ROUNDING AND COMPENSATION

71 x 27	(Balancing by 3)	31 x 53	(Rounding and Compensation)
344 x 44	(Balancing by 4)	62 x 740	(Rounding and Compensation)
25 x 70	(Balancing by 5)	101 x 341	(Rounding and Compensation)
444 x 36	(Balancing by 6)	97 x 432	(Rounding and Compensation)
81 x 882	(Balancing by 9)	9 x 999	(Rounding and Compensation)
121 x 5,159	(Balancing by 11)		
8 x 583	(Rounding and Compensation)		

ASSOCIATING AND DISTRIBUTING

43 x 444	(Associative)	35 x 858	(Associative)
312 x 45	(Associative)	53 x 25	(Distributing)
382 x 820	(Associative)	712 x 63	(Distributing)
84 x 583	(Associative)	49 x 17	(Distributing)
997 x 24	(Associative)	861 x 34	(Distributing)
663 x 58	(Associative)	77 x 96	(Distributing)

950 x 81 (Distributing)

TRIPLE DIGIT PRODUCTS

212 x 485 (Three digits by three digits)

667 x 193 (Three digits by three digits)

230 x 484 (Three digits by three digits)

431 x 345 (Three digits by three digits)

485 x 950 (Three digits by three digits)

PRODUCTS WITH MULTIPLICANDS CLOSE TO POWERS OF 10

375 x 99 (One away from 10)

462 x 999 (One away from 10)

891 x 9,999 (One away from 10)

97 x 98 (A few away from 10 on both multiplier and multiplicand)

998 x 999 (A few away from 10 on both multiplier and multiplicand)

9,998 x 9,997 (A few away from 10 on both multiplier and multiplicand)

CHAPTER 4: DIVIDING FAST PART I

In this chapter we'll cover some fundamental concepts to perform quick in your feet mental divisions.

We'll cover some basic rules of divisibility that you may have learned in school and put them in the context of fast mental calculations.

DIVISION AS THE INVERSE OF MULTIPLICATION

If you have your basic multiplication tables down, but division takes more work, then this essential fact might come in very handy.

Division is the *inverse of multiplication*.

An example,

3 x 4 = **12**.
What is the result of **12** / 3?

Indeed, it is 4.

When the ***dividend*** **(12)** in a division is equal to the ***product*** **(12)** in a multiplication, if the *divisor* is equal to the *multiplicand,* and the ***quotient*** is equal to the multiplicator. *(Remember that multiplicator and multiplicand are interchangeable)*

The theoretical definition will be easier to grasp with some examples.

2 x 3 = **6**	So, **6** / 3 = 2	& **6** / 2 = 3
4 x 5 = **20**	So, **20** / 5 = 4	& **20** / 4 = 5
8 x 7 = **56**	So, **56** / 7 = 8	& **56** / 8 = 7

This means effectively when you are dealing with a division, if you know how to multiply fast and have memorized some of the basic single and double-digit products, that you can reverse engineer the division to get the number by asking yourself one question:

What do I need to multiply the divisor by to get the dividend?

So, in **20** / 4… you would ask yourself, 4 times what is **20**?
And, in **20** / 5 you would ask, 5 times what is **20**?

Whenever you are trying to solve a hard division where the divisor is a multiple of 3 (which you can check by adding the digit place values and easier to recognize multiple of 3, use reverse engineering to solve the problem.

Note for example, 342 is a multiple of 3, we determine this because 3 + 4 + 2 is 9, which is 3 x 3.

Easier to recognize multiples of 3 are, 3, 6 and 9.

If you get a number like 3,414, you see that 3 + 4 + 1 + 4 (called *the digital sum*) is 12, take the digital sum of 12, add 1 + 2, is 3, which like 6 and 9, is a multiple of 3.

Example 1	35 / 7 7 x 5 = 35 5

Example 2	121 / 10 10 x 12.1 = 121 12.1

Example 3	72 / 6 6 x 12 = 72 12

ADVANCED HALVING

Many of the techniques for fast multiplication reversed, can be used to calculate quotients fast.

Doubling for multiplication has its mirror technique in ***halving*** for division.

100 / 2 = 50. ***Halving*** is nothing more than dividing by 2. Following that line of reasoning divisions by 4, can be thought of as ***halving*** twice, and divisions by 6 as ***halving*** three times and so on and so forth for all powers of 2.

1,460 / 8 would require you to think of ***halving*** three times instead of the traditional long notation algorithm. Once you learn to do this quickly, computing any division by 4, 8, 16, or whatever whole power of two will be easier.

1,460 halved, is 730, which halved is 365, which halved is 182.5. So, 1,480 / 8 = 182.5.

It's a good idea to practice this technique ***halving*** three times to acquire fluency dividing by 2. But ***halving*** would find its most efficient use as a go to complement of the other techniques we'll introduce.

And, as we will see later-on, it will come in handy also when dividing by 20, 40, 80, 160, 200, and so on.

Example 1	3200 / 8 1600, 800, 400 400

Example 2	640 / 4 320, 160

	160

Example 3	3760 / 16 1880, 940, 470, 235 235

DIVISIBILITY BY THREE, NINE, FIVE AND TEN.

To Determine if a number is divisible by 3 or 9, simply add all digit place values, if the resulting number (which may be more than 1 digit) is divisible by 3 or 9. All numbers divisible by 9 are divisible by 3 by the way, since 9 is a multiple of 3.

Examples

27 = 2 + 7 = 9…. It is divisible by both 3 and 9.
30 = 3 + 0 = 3… It is divisible by 3 but not 9.
275 = 2 + 7 + 5 = 14 which is divisible by neither… you can further try 1 + 4 = 5 to check for divisibility. Since 5 is not divisible by 3 or 9, it is not.

2,754 = 2 + 7 + 5 + 4 = 18, 1 + 8 = 9, it is divisible by both 3 and 9.

Five and ten share a similar relationship in that if a number is divisible by 10 it is divisible by 5 (and by 2 since 10 is even, and all even numbers are divisible by 2.

For a number to be divisible by 5 its last place value digit needs to end in 5 or 0.
For a number to be divisible by 10 its last place value digit needs to end in 0.

Example 1	495 Digital sum = 4 + 9 + 5= 18 Digital sum = 1 + 8 = 9 Divisible by both 3 and 9

Example 2	241 Digital sum = 2 + 4+ 1 = 7 Not divisible by 3 or 9

Example 3	351 Digital sum = 3 + 5 + 1 = 9 Divisible by both 3 and 9

DIVISIBILITY BY EIGHT AND FOUR

To know if a number is divisible by 8, you need to look at the rightmost *last three place values*, the *hundreds, tens, and ones*. If the last three digits taken as a number are divisible by 8 so is the entire number to which those figures are prepended.

This is the standard advice given in school. Effectively it means that you need to know whether 125 numbers are divisible by 8. That's not very practical…

For instance, let's see the first 100 numbers we need to know according to traditional teachings to see if we can further simplify it:

008 016 024 032 040	208 216 224 232 240	408 416 424 432 440	608 616 624 632 640
048 056 064 072 080	248 256 264 272 280	448 456 464 472 480	648 656 664 672 680
088 096 104 112 120	288 296 304 312 320	488 496 504 512 520	688 696 704 712 720
128 136 144 152 160	328 336 344 352 360	528 536 544 552 560	728 736 744 752 760
168 176 184 192 200	368 376 384 392 400	568 576 584 592 600	768 776 784 792 800

856 is divisible by 8 simply by noting that 800 (from the fourth group of numbers) + 056 from the first group of numbers is divisible by 8. So, the sum of the fourth group plus a number in the first group will result in a number divisible by 8 also.

But let's make it even simpler. Note that we can further simplify this list by splitting numbers into two groups, groups where the hundreds place value is even, and numbers where hundreds place numbers is odd.

E08	E16	E24	E32	E40
E48	E56	E64	E72	E80
E88	E96	O04	O12	O20
O28	O36	O44	O52	O60
O68	O76	O84	O92	E00

Now the numbers we need remember are just 25, split into groups of even hundred place value numbers, and odd hundred place value numbers.

If a number is divisible by 8 it must be divisible by 4 since 8 is a multiple of 4.

And just as a number may be divisible by 2 but not 4, or by 2 but not 6, or by 2 but not 10, or by 5 but not 10, and so on… a number may also be divisible by 4 but not by 8

If the last two digits of the number are divisible by 4 so is the entire number; knowing the table of four and up to twenty-five is enough know if any number at all is divisible by 4.

04	**08**	**12**	**16**	**20**
24	**28**	**32**	**36**	**40**
44	**48**	**52**	**56**	**60**
64	**68**	**72**	**76**	**80**
84	**88**	**92**	**96**	**100**

Which, along the same lines, could be further simplified, so as not to remember 25 numbers but just 5 combinations. Looking at any numbers last two digits, if the tens place value is even, and the ones is 0, 4, or 8, the number is divisible by four, if the tens place value is odd, and the ones is 2 or 6, it also is divisible by 4, otherwise it is not.

E0 O2 E4 O6 E8

Example 1	4282 282 = E82 = Not divisible by 8 82 = E2 = Not divisible by 4

Example 2	1488 488 = E88 = Divisible by 8 88 = E8 = Divisible by 4

Example 3	1428 428 = E28 = Not divisible by 8 28 = E8 = Divisible by 4

DIVISIBILITY BY SIX

If a number is *even* and if it is *divisible by 3*, then it is *divisible by 6*. And we already saw that to be divisible by 3, you simply add the place value digits and determine it the resulting number is a multiple of 3. If it is many digits and that's hard to figure out simply keep adding the digits.

Example

1,047 Since the number is not even we don't even go further. It cannot by divisible by 6.

1,046 is even. Next, calculate the digital sum
1 + 0 + 4 + 6 = 11, 1 + 1 = 2.
Which is not divisible by 3, therefore, it cannot be divisible by 6.

3,756 is even. Next, calculate the digital sum
3 + 7 + 5 + 6 = 21, 2 + 1 = 3
although it should be apparent that 21 is divisible by 3, since 3x7 = 21.
So, 3756 is divisible by 6.

Example 1	3482 3 + 4 + 8 + 2 = 16, 1 + 6 = 7 Even, but not divisible by 3 (or 6)

Example 2	582 5 + 8 + 2 = 15, 1 + 5 = 6 Even, and divisible by 3, and therefore by 6

Example 3	483 4 + 8 + 3 = 15, 1 + 5 = 6 Odd, and divisible by 3, and therefore, not divisible by 6

DIVIDING BY SEVEN

When dividing by seven there is a special technique that will come a long way.

First, notice something curious (note we are not rounding the 6th decimal place, just truncating it)

1/7 = **0.142857**
5/7 = **0.714285**
4/7 = **0.571428**
6/7 = **0.857142**
2/7 = **0.285714**
3/7 = **0.428571**

When you have a division such as 73 / 7… you can easily decompose that into 70 / 7 + 3 / 7… since we know that 70 / 7 is 10. We can pinpoint the exact number by simply adding the decimal places **0.428571** from the chart to get 10.428571

we can break down whatever two, three our four-digit division we have by 7 and get the exact few first decimal places with this rainbow chart.

So, if you had to calculate 65 / 7, we would break it into 7 x 9 = 63, so 65 / 7 can be expressed as 63 /7 + 2/7, or 9.285714

While that's all neat, you may still have problems figuring out how to arrive at the whole number to which you should add the fraction of 7 to begin with, especially with three-digit figures.

Easily enough, there's a rule, which simply consists of taking the ones place value digit of the dividend, multiplying it by two and subtracting it from the leftmost place value digits. If the resulting number is divisible by 7, then you have a whole number to work with.

For instance,

347 / 7
7 x 2 = 14
34 - 14 is **20**, which is not divisible by seven.

456 / 7
6 x 2 = 12
45 - 12 is **32**, which is not divisible by seven.

875 / 7

5 x 2 = 10
87 - 10 is **77**, which IS divisible by seven.
7 x 11 is 77 which is a multiple of 7

When dealing with three-digit numbers… knowing single-digit multiplication by 7 is enough to figure out any division by 7.

If you so desire, you can memorize the decimal places represented by any single or double digit denominator where the numerator is less than the denominator to calculate with whatever precision you desire, the exact number of absolutely any division.

So for instance, if you know that **¼ is 0.25, 2/4 is 0.5** and **3/4 is 0.75**.

You can easily determine that
37 / 4 = 9**.25**, because
4 x 9 is 36 (one less than the divisor, the 1 in 1/4)
9 + **¼** is 37 / 4

or
85 / 4 = 21.**25**, because
4 x 21 is 84 (one less than the divisor, the 1 in 1/4)
21 + **¼** is 85 / 4

or
74 / 4 = 18.**5**, because
4 x 18 = 72 (two less than the divisor, the 2 in 2/4)
18 + **2/4** is 74 /4

and on and so forth,

Here's a brief table for reference, rounded to the nearest hundredth, should you decide this is worthwhile. That is, if you need to figure out the first two decimal places in a division.

0 .1 .2 .3 .4 .5 .6 .7 .8 .9 1

1/2 .50

1/3 .33 2/3 .67

1/4 .25 2/4 .50 3/4 .75

1/5 .20 2/5 .40 3/5 .60 4/5 .80

1/6 .17 2/6 .33 3/6 .50 4/6 .67 5/6 .83

1/7 .14 2/7 .29 3/7 .43 4/7 .57 5/7 .71 6/7 .86

1/8 .13 2/8 .25 3/8 .38 4/8 .50 5/8 .63 6/8 .75 7/8 .88

1/9 .11 2/9 .22 3/9 .33 4/9 .44 5/9 .56 6/9 .67 7/9 .78 8/9 .89

Example 1	48 / 7 42/7 + 6/7 6.857142

Example 2	282 / 4 280/4 + 2/4 70.5

Example 3	386 / 5 385 / 5 + 1 / 5 77.2

REVERSE ENGINEER THE RULE OF 11

Because division is the inverse of multiplication we can reverse engineer the ***rule of multiplying by 11*** to recognize easily figures whose result is 11.

253 for example is clearly 23 x 11 and 484 clearly 44 x 11. (If this is not evident, review the ***rule of multiplying by 11***)

So, when we have a three-digit division where the dividend is arranged such that the dividend's hundred place value plus the ones place value results in the tens place value figure, clearly 11 is the quotient.

253 / 23, or 484 / 44 are such examples.

Moving along those lines understanding how carries work with the ***rule of multiplying 11*** for multiplication we can also see that

319 / 29 is also 11, as is 407 / 37.

Just recognize that the ones digit place value must be the same in the dividend and divisor (here 407 and 37).

If the digital sum of the divisor results in a two-digit figure then the ones of that figure will be equal to the tens value in the dividend, and the hundreds place value in the dividend will be one more than the tens place value figure in the divisor.

Again, practicing will make it more apparent. Here are some divisions which yield 11 as a quotient

748 / 68	check the ones	6+8 = 14	the tens	7 = 6+1	& the hundreds
506 / 46	check the ones	4+6 = 10	the tens	5 = 4+1	& the hundreds
858 / 78	check the ones	7+8 = 15	the tens	8 = 7+1	& the hundreds

Example 1	891 / 81 89**1** / 8**1** 8+1 **= 0**9 8 = 8 + **0** 11

Example 2	495 / 45 49**5** / 4**5** 4+5 **= 0**9 4 = 4 + **0** 11

Example 3	803 / 73 80**3** / 7**3** 7+3 **= 1**0 8 = 7 + **1** 11

ROUNDING AND COMPENSATION FOR DIVISION

If you divide a number by 2, that is you ***halve*** a number like 476 or 388, where at least one of the digits is odd, it's good to resort to ro***unding and compensation*** to simplify things.

Why might ***Halving*** 476 be a problem? Half of 400 is 200. That doesn't require much thought. But remembering that half of 70 is 35 and then remembering that half of 6 is 3 to get a total of 238 might be a bit of a stretch. This might be even more complicated where the hundreds and tens are odd numbers.

Sticking to our example, simply round 476 to 500, find half which is 250, and subtract half of the units you added to round it up (remember half the units because we halved the original amount, so compensation must be proportional). In this case, half of 24 is 12 so 250 - 12 is 238, which is quicker.

Equally 388 is 150 + 40 + 4 = 194, not too hard maybe. But what about the more efficient, half of 400 is 200, 200 - 6 = 194.

If you are rounding a number where all digits are even 286, just go from left to right, and halve each individual digit to get the result quicker, half of 28**6** is 14**3**.

Finally, if you are rounding down say 516 to 500, when halving, you wouldn't subtract, but add half the units instead. So, 250 + 8 = 258

The rules remain the same:

if you are rounding up a positive number, you must ***compensate by subtracting***,
if you are rounding down a positive number, you must ***compensate by adding***.

Example 1	398 / 2 400 / 2 = 200, 200 - 2 / 2 199

Example 2	810 / 2 800 / 2 = 400, 400 + 10 / 2 405

Example 3	9,184 / 2 9,000 / 2 = 4,500, 4,500 + 184 / 2 4,500 + 92 4,592

ASSOCIATIVE PROPERTY OF DIVISION

Like the method presented in multiplication, you can simplify division by breaking down one of the terms into more but easy to handle divisions such as those where the divisor is 2, 3, 5 or 10.

396 / 18 is really 396 / 2 / 9 = 396 / 2 / 3 / 3

and

345 / 15 is really 345 / 3 / 5

396 / 2 / 3 / 3
400 / 2 = 200, 200 - (4 / 2) = 198
198 / 3
    19 / 3 = **6** %1
    18 / 3 = **6** %0
    66
66 / 3
    6 / 3 = **2** %0
    6 / 3 = **2** %0
22

and

345 / 3 / 5 is
345 / 5 / 3

although indistinct, we switch the terms, since dividing by a larger number may be useful as the resulting figure from the first division, will presumably be smaller.

There are arguments for and against dividing 3 or 5 first, it depends ultimately to you.

345 / 5
    34 / 5 = **6** %4
    45 / 5 = **9** %0
69

69 / 3
    6 / 3 = **2** %0
    9 / 3 = **3** %0
23

Remember we are working with whole numbers through this series. We will consider decimals for divisions since unlike addition, subtraction, and multiplication, in the case of division even when all your operands are whole numbers, you can still get decimals in the result. We'll cover that scenario later on.

Example 1	903 / 21 903 / 7 / 3 903 / 7 9 / 7 = 1%2 20 / 7 = 2%6 63 / 7 = 9%0 129 / 3 12 / 3 = 4%0 9 / 3 = 3%0 43

Example 2	2,232 / 24 2,232 / 6 / 4 2,232 / 6 22 / 6 = 3%4 43 / 6 = 7%1 12 / 6 = 2%0 372 / 4 37 / 4 = 9%1 12 / 4 = 3%0 93

Example 3	3,465 / 45 3,465 / 9 / 5 3,465 / 9 34 / 9 = 3%7 76 / 9 = 8%4 45 / 9 = 5%0 385 / 5 38 / 5 = 7%3 35 / 5 = 7%0 77

DISTRIBUTING A TERM FOR DIVISION

Similarly, if you are dealing with divisions where the divisor is 2, 3, 5, 10, or any other number that comes easy to you, and the dividend is a large figure… you may distribute the dividend to make it more manageable, like so,

1,495 / 5
is the same as
1,000 / 5 + 495 / 5
1,000 / 5 = 200
If you didn't recognize this immediately, remember that 5 x 10 is 50 so 5 x 100 is 500, double 100 it is only natural that 5 x 200 is 1000.

495 / 5

500 / 5 = 100, 100 - 5/5

99

200 + 99 is the answer.

To check, reverse engineer the operation, calculating whether the result times the divisor equals the dividend.

299 x 5

300 x 5 = 1500 - 1 x 5

1495

Another example that lends itself to this technique might be the following

1,263 / 3

It's both easy to see that 1200 and 63 as separate terms are divisible by 3, so simply do

1,200 / 3, which is 400 (because 3 x 4 = 12), and add 63 / 3, which is 21, (because 3 x 2 = 6).

400 + 21

421

Example 1	4,815 / 5 4,000 / 5 + 815 /5 800 + 800 / 5 + 15 / 5 800 + 160 + 3 963

Example 2	8,280 / 9 8,100 / 9 + 180 / 9 900 + 20 920

Example 3	38,208 / 2 38,000 / 2 + 208 / 2 19,000 + 104 19,104

CHAPTER 5: DIVIDING FAST PART II

Chapter 5 is devoted to presenting general techniques for division based on the principles and particular cases developed in Chapter 4. This is mental division at its core.

BALANCING FOR DIVISION WITH PRIMES

We can adapt the ***balancing technique*** learned for multiplication when dividing multiples of prime numbers, especially large ones.

We'll first give an example with a common prime, 3.

Recall ***balancing***,

39 x 18 = (39 **x 3**) x (18 **/ 3**) = 117 x 6 = **702**

We can do something similar in division, only this time we will divide both terms, but only when the dividend shares a multiple/divisor relationship with the other term. Thus,

132 / 22 = (132 **/ 11**) / (22 **/ 11**) = 12 / 2 = **6**

In the above example 132 / 22, 132 is a multiple of 22 and 22 a divisor of 132.

So seeing 12 / 2 is much easier than calculating 132 / 22.

To use this technique, you need to have acquired a degree of familiarity with numbers.

Alternatively, you could have halved the numbers since they are even,

132 to 66 and 22 to 11, thus getting 66 / 11, which would have been irreducible since 11 is a prime number. The quotient being again, **6**.

Let's see another example,

52 / 26, halve both
26 / 13, which is
2

or

52 / 26 = (52 / 13) / (26 / 13) = 4/2 = **2**

It's worth noting that simplifying or reducing a division typically focuses on numbers that are divisible by other numbers.

But what about primes?

Let's recall the first few primes:

2 3 5 7 11 13 17 19
23 29 31 37
41 43 47 53 59
61 67 71 73 79
83 89 97

When you come to a quotient with a divisor like 127, you can immediately round it up as a suspect, This number is clearly not even, so it is not divisible by 2, 4, 6, 8 or 10. It is also clear that this number is not divisible by 5 and quick digital sum reveals it is not divisible by 3 or 9. You may be tempted to spend a while trying to figure out whether the number is divisible by 7 or 9 to see if you can further reduce it. But this number is only divisible by itself. To avoid wasting your time, and to simplify as much as you can, remember not only the two-digit primes, but a few three-digit primes as well:

101 103 107 109 113 127 131 137 139
211 223 227 229 233 239
307 311 313 317 331 337
401 409 419 421 431 433 439
503 509 521 523
601 607 613 617 619 631
701 709 719 727 733 739
809 811 821 823 827 829 839
907 911 919 929 937

149 151 157 163 167 173 179 181 191 193 197 199
241 251 257 263 269 271 277 281 283 293
347 349 353 359 367 373 379 383 389 397
443 449 457 461 463 467 479 487 491 499
541 547 557 563 569 571 577 587 593 599
641 643 647 653 659 661 673 677 683 691
743 751 757 761 769 773 787 797
853 857 859 863 877 881 883 887
941 947 953 967 971 977 983 991 997

Remember these numbers can't be simplified further, so you must solve the division as is.

Example 1	210 / 30 (210 / 3) / (30 / 3) 70 / 10 = 7

Example 2	210 / 30 = (210 / 30) / (30 / 30) = 7 / 1 = 7

Example 3	376 / 47 = (376 / 47) / (47 / 47) = 8 / 1 = 47

QUICK METHOD FOR LARGE DIVIDENDS & SMALL DIVISORS

When you are dividing whole numbers by a single digit divisor, no matter how large the dividend may be, so long you are familiar with single and double-digit multiplication tables, and basic quick addition and subtraction, you can calculate the quotient in no time. Some examples,

Take the following division,

930 / **6**

6 into **9** is **1**, reminder 3, attach to next digit in line making 3**3**
6 into 3**3** is **5**, reminder 3, attach to next digit in line making 3**0**
6 into 3**0** is **5**, reminder 0.

155

We've seen this previously as:

930 / 6
9 / 6 = **1**%3
33 / 6 = **5**%3
30 / 6 = **5**%0

155

Which might be a better way to imagine the process when calculating mentally.

329 / **7**

7 into 3 doesn't fit so, attach to the next digit in line make 32
7 into **32** is **4**, reminder 4, attach to next digit in line making 4**9**
7 into 4**9** is **7**, reminder 0.

result **47**

3,485 / **5**

5 into 3 doesn't fit so, attach to the next digit in line, make 34
5 into **34** is **6**, remainder 4, attach to the next digit in line make 4**8**
5 into 4**8** is **9**, remainder 3, attach to the next digit in line make 3**5**
5 into 3**5** is **7**, remainder 0

result **697**

QUICK METHOD FOR LARGE DIVIDENDS & TWO DIGIT DIVISORS

Consider the following problems and the method. We focus on whole numbe quotients to simplify and make the method algorithm apparent, although most divisions will result in a fraction or decimal number.

165 / **55**

55 into **165** is at most ***3***, reminder 0.

result ***3***

748/**34**

34 into **74** is at most ***2***, reminder 6, attach to next digit in line **8**
34 into **68** is at most ***2***, reminder 0.

result ***22***

798 / **14**
14 into **79** is at most ***5***, reminder 9, attach to the next digit in line make **98**
14 into **98** is at most ***7***, reminder 0.

result ***57***

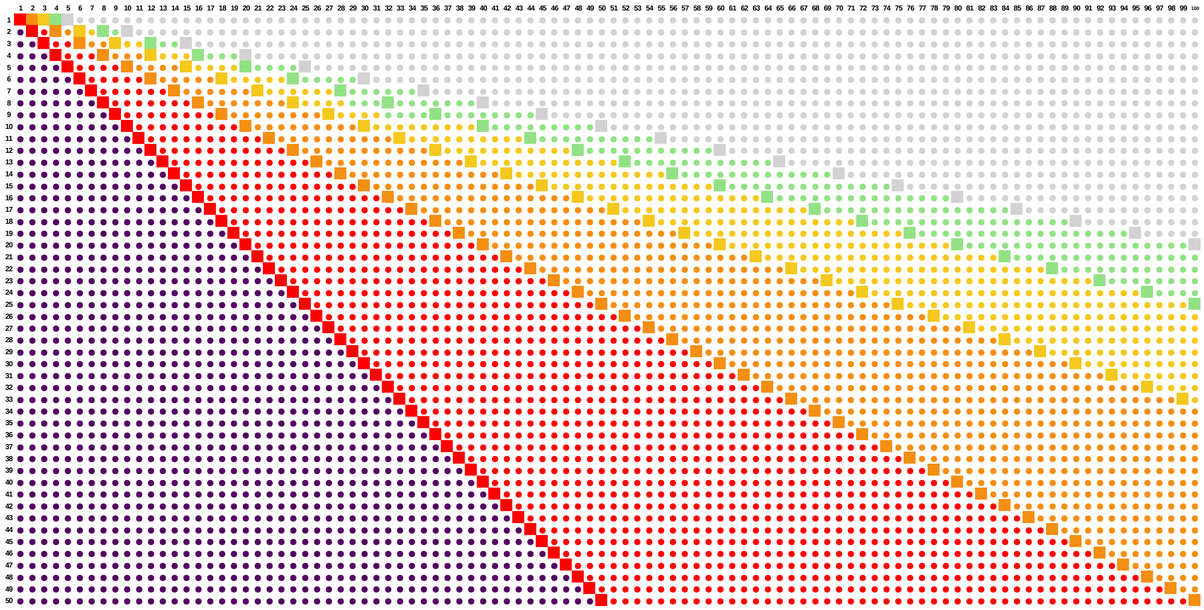
This method may appear difficult at first glance, but in all honesty, it is not.

It does however require two essential skills you must acquire to tackle these and larger digit division.

These are

- One, finding how many times a two-digit figure fits into another two-digit figure, and
- Two, knowing how to calculate one and two-digit reminders via fast addition and subtraction.

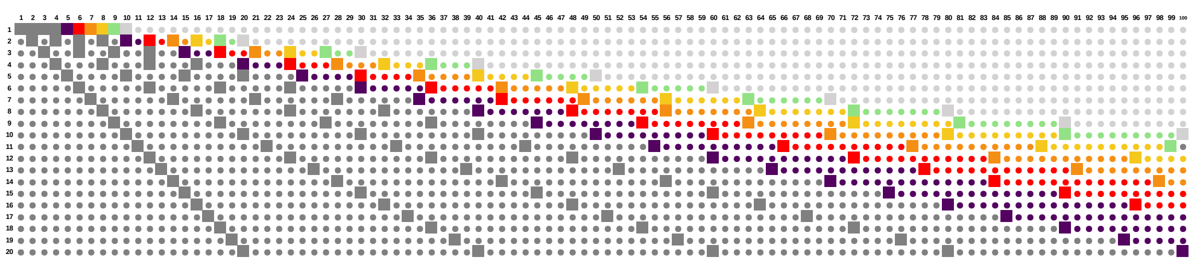
Acquiring the first skill is not as dauting as it may seem. The following visual aid will assist you in making this easier. Just analyze the chart carefully, reading it as follows.



The first column numbers (each row) represent the divisor, the first-row numbers (each column) the number you are trying to fit the divisor into.

For example, the intersection of the first row and the first column falls into the grey area (0x) or zero times, which is how many times 10, fits into 1. Moving to the intersection off the first row and the 45th column, we fall into the pink area (4x) which is how many times 10 fits into 45.

The black markings on the image represent for that given combination, how many times the divisor fits into the two-digit number in question.



Close inspection reveals at least two meaningful and useful facts.

The first is that we should only worry about divisors (in the first column) that are 50 or less than 50 for the simple fact that divisors greater than 50 will either not fit into the two-digit number you are trying or will fit only once, which will be in any case, easy to realize. (See how the light grey and light green would sweep all the divisors greater than 50?

The second is that although it would be useful for you to familiarize yourself with this image, in practice, the only divisors that are hard to compute are those from 10 to 19. Why?

Because they are the smallest figures and the ones that can fit at most into other two digit figures the most times. Focus on those, that is at the top, and particularly, at the top-right corner of the image, that's where things get trickier.

In other words, how many times 25 fits into 70 is not something difficult to figure out. On the other hand, how many times 17 fits into 80, is significantly more difficult. And again, this last combination, falls into the top right corner of the graph. (By the way it fits no more than 4 times, although it gets close to 5), so it falls in the pink area.

Example 1	646 / 38 64 / 38 = 1 %26 266 / 38 = 7 %0 17

Example 2	5720 / 40 572 / 4 5 / 4 = 1%1 17 / 4 = 4%1 12 / 4 = 3%0 143

Example 3	780 / 39 78 / 39 = 2 %0 0 / 39 = 0 %0 20

DIVISOR LARGER THAN DIVIDEND

So, you've come all this way, and just as you might have wondered what if the quotient is not a whole number? You might rightfully ask, what if the divisor is larger than the dividend.

Well in that case, as we've seen with single digit fractions, to the left of the decimal mark you'll have 0. Then you'll add a 0 to the dividend and proceed with the operation. Still having a larger divisor than dividend, add a zero to the right of the decimal places, append a 0 in the ones place value in the dividend and divide using the methods we've covered.

For instance, given

3 / 35

Compute

30 / 35

And your answer will start with
0. something

But 30 is still smaller than 35. No problem. Now we know, your answer will start with
0.0…something

35 fits **8** times in 300. If this is hard to see, practice!

Again, 35 fits **8** times in 300, with 20 units remaining. That means that 280 / 35 is **8** exactly.

We arrive at 280 by subtracting 20 from 300.

So far **0.08** plus 20/35 which can be reduced to **4/7**, or **0.57** so **0.0857** is the answer.

Let's do another one.

1 / 75, do 100 / 75, with **0.0** in the answer, 75 / 75 is 1, so **0.01** appending…. 25 / 75, which we could think as 1/3 since 25 fits exactly 3 times into 75.

So, a good guess would be **0.013**

34 / 390 will be our final example

340 / 390 is 0., and 3400 / 390 is **0.0**, now divide the fraction by 10. 340 / 39 to simplify things. 39 fits

about **8** times in 340 with a reminder of 28, so **0.08** appending 28 / 39, which is a little close to ¾ or .75 (but not quite). So, we have **0.087**

As you are breaking down fractions to get the rightmost decimal place values by figuring out how to simplify or approximate fractions, recall there are some numbers that can't be broken down further. These are the primes, divisible only by themselves and 1. Commit the first primes to memory.

2 3 5 7 11 13 17 19
23 29 31 37
41 43 47 53 59
61 67 71 73 79
83 89 97

Example 1	44 / 921
0.	
0.0	440 / 921
0.04	4400 / 921
0.047	4400 / 921 = 4%716
0.0477	7160 / 921 = 7%713
0.0478	7130 / 921 = 7%683
	remainder is half or more than half the divisor, round up

Example 2	179 / 284
0.	
0.6	1790 / 284
0.63	1790 / 284 = 6%86
0.630	860 / 284 = 3%8
0.6302	80 / 284 =
0.6303	800 / 284 = 2%232
	remainder is half or more than half the divisor round up

Example 3	
	9 / 128
0.	
0.0	90 / 128
0.07	900 / 128
0.070	900 / 128 = 7%4
0.0703	40 / 128 =
0.0703	400 / 128 = 3%16
	remainder is less than half the divisor, don’t round up

CHAPTER 6: PROGRESSIVE PRACTISE II

We continue practicing mental arithmetic but we turn our attention now to division.

We move forward gradually. Again, I have given a suggested way of solving each exercise in case you get stuck.

There are hundreds of thousands different quotients so keep practicing beyond and trying to apply the steps we presented previously to your own examples.

Also make sure to go through the Appendix as it has worked out answers that correspond with the suggestions we present here on how to solve the problems.

SINGLE DIGIT DIVISORS

473 / 4	(advanced halving)	444 / 8	(dividing by 8 and 4)
581 / 8	(advanced halving)	3,808 / 4	(dividing by 8 and 4)
4,560 / 8	(advanced halving)	4,796 / 8	(dividing by 8 and 4)
5,738 / 4	(advanced halving)	384 / 6	(dividing by 6)
474 / 3	(dividing by 3 and 9)	999 / 6	(dividing by 6)
219 / 9	(dividing by 3 and 9)	747 / 6	(dividing by 6)
1,311 / 3	(dividing by 3 and 9)	479 / 7	(divisibility by 7)
3,819 / 9	(dividing by 3 and 9)	123 / 7	(divisibility by 7)
316 / 4	(dividing by 8 and 4)	282 / 7	(divisibility by 7)

RULE OF 11 AND ROUNDING AND COMPENSATION

242 / 11	(reverse engineer rule of 11)	1,157 / 6	(rounding and compensation)
847 / 11	(reverse engineer rule of 11)	2,678 / 3	(rounding and compensation)
1,089 / 11	(reverse engineer rule of 11)	1,421 / 7	(rounding and compensation)
489 / 4	(rounding and compensation)	5,589 / 5	(rounding and compensation)

ASSOCIATING AND DISTRIBUTING IN DIVISION

105 / 15	(associative property)	749 / 7	(distributive property)
516 / 12	(associative property)	621 / 3	(distributive property)
2,130 / 30	(associative property)	1,015 / 5	(distributive property)
1,584 / 16	(associative property)	4,998 / 49	(distributive property)
4,956 / 21	(associative property)	14,271/ 71	(distributive property)
49,728 / 64	(associative property)	9,797 / 97	(distributive property)

BALANCING

117 / 39	(balance with primes)	588 / 294	(balance with primes)
171 / 57	(balance with primes)	70 / 14	(balance)
1,616 / 404	(balance with primes)	420 / 64	(balance)

LARGE DIVIDENDS

4,635 / 3	(large dividends, one-digit divisors)
4,995 / 5	(large dividends, one-digit divisors)
522/ 6	(large dividends, one-digit divisors)
2,928 / 4	(large dividends, one-digit divisors)
21,792 / 93	(large dividends \ two-digit divisors)
4,992 / 48	(large dividends two-digit divisors)
6,534 / 66	(large dividends two-digit divisors)
5,929 / 77	(large dividends two-digit divisors)
4,575 / 61	(large dividends two-digit divisors)
40,131 / 63	(large dividends two-digit divisors)
48,840 / 88	(large dividends two-digit divisors)
10,248 / 21	(large dividends two-digit divisors)
12,845 / 55	(large dividends two-digit divisors)
48,179 / 18	(large dividends two-digit divisors)
10,384 / 32	(large dividends two-digit divisors)

DIVISORS LARGER THAN DIVIDENDS

1 / 99 (divisor larger than dividends)

3 / 47 (divisor larger than dividends)

5 / 36 (divisor larger than dividends)

7 / 32 (divisor larger than dividends)

6 / 49 (divisor larger than dividends)

3 / 481 (divisor larger than dividends)

5 / 783 (divisor larger than dividends)

9 / 2,903 (divisor larger than dividends)

10 / 3,841 (divisor larger than dividends)

15 / 1,374 (divisor larger than dividends)

19/ 40,100 (divisor larger than dividends)

203 / 425 (divisor larger than dividends)

594 / 671 (divisor larger than dividends)

11 / 1,111 (divisor larger than dividends)

99 / 3,847 (divisor larger than dividends)

23 / 2,323 (divisor larger than dividends)

10 / 9,999 (divisor larger than dividends)

4,839 / 14,748 (divisor larger than dividends)

2,244 / 3,399 (divisor larger than dividends)

5,837 / 9,891 (divisor larger than dividends)

5,783 / 5,863 (divisor larger than dividends)

11 / 121 (divisor larger than dividends)

17 / 34,747 (divisor larger than dividends)

9,999 / 99,999 (divisor larger than dividends)

999 / 1,000 (divisor larger than dividends)

PARTING WORDS

Thank you for reading this book. I really do hope you found it useful and enjoyable. Math is challenging, that's what makes it fun.

If you review and re-do the exercises in the practice chapters, make sure you also review the appendix.
Did you find a better way to arrive at the result? Congratulations!
Success is all about taking matters into your hands!

Did you like the book? Let us know, we love to hear what we did well. It encourages us to pursue greater challenges.

Did you find things you didn't like? Do let us know also. It's only human to make mistakes, however painstakingly we may have reviewed this book before it was published.

In any case, I wish you the best in your journey to achieving mastery in this or any other areas of your life.

And I hope that you will join us as we explore the world of mathematics in future publications.

With warm regards, all the best!

The Sigmacasts Team

APPENDIX A1: WORKED OUT ANSWERS TO PROBLEMS (CHAPTER 3)

PRODUCTS TO MEMORIZE

7x6

6x7

42

5x9

45

8x7

7 14 28

56

9x8

9 18 36

72

9x6

54

9x7

7x9

63

11 x 17

170 + 17

187

12 x 13

156

13 x 14

140 + 14 x 3

182

14 x 15

1
140 + 140 / 2 =

210

15 x 16

160 + 160 / 2 =

240

16 x 17

170 + 17x6

272

17 x 18

180 +18x7

306

18 x 19

190 x 19x8

342

11 x 14

140 + 14x1

154

12 x 19

190 + 19 x 2

228

13 x 18

180 x 18 x 3

234

14 x 17

170 x 17x4

238

DOUBLING AND MULTIPLES OF 10

243 x 2	**17 x 4**	**479 x 4**	**19x80**	**9 x 150**	**5 x 77**
486	17 34	479 858	190 380 760	90 x 15	770 / 2
	68	**1,916**	**1,520**	900 + 900 / 2	**385**
				1,350	
384 x 2	**38 x 8**	**13 x 8**	**7 x 400**	**5 x 41**	**5 x 780**
768	38 76 152	13 26 52	**2,800**	410 / 2	7800 / 2
	304	**104**		**205**	**3,900**
562 x 2	**139 x 4**	**79 x 4**	**7 x 340**	**5 x 180**	**5 x 950**
1,124	139 278	79 158	**2,380**	**1800 / 2**	9500 / 2
	556	**316**		**900**	**4,750**
719 x 2	**385 x 8**	79 x 8	**3 x 200**		
1,438	385 770 1540	79 158 316	**600**		
	3,080	**632**			

SQUARES ENDING IN FIVE, AND PRODUCTS WITH ELEVEN

35 x 35	**75 x 75**	**11 x 733**
3x(3+1), 25	7x(7+1), 25	**8,063**
1,225	**5,625**	
15 x 15	**11 x 8**	**11 x 89**
1x(1+1), 25	**88**	**979**
225		
155 x 155	**11 x 313**	**11 x 220**
15x(15+1), 25	**3443**	**2,420**
24,025		
85 x 85	**11 x 54**	**11 x 35**
8x(8+1), 25	**594**	**385**
7,225		

RULE OF FIFTEEN AND TWO DIGIT FIGURE PRODUCTS

15 x 83

830 + 830 / 2
830 + 415

1,245

15 x 731

7,310 + 7,310 / 2
7,310 + 3,500 + 150 + 5
7,310 + 3,655

10,965

67 x 81

6x8 = 48
7x1 = 7
6x1 + 7x8 = 62

48 62 7

5,427

15 x 120

1,200 + 1,200 / 2
1,200 + 600

1,800

94 x 14

9x1 = 9
4x4 = 16
4x9 + 1x4 = 40

9 40 16

1,316

15 x 35

1x3 = 3
5x5 = 25
1x5 + 5x3 = 20

3 25 20

525

15 x 165

1,650 + 1,650 / 2
1,650 + 825

2,475

34 x 52

3x5 = 15
4x2 = 8
3x2 + 4x5 = 26

15 26 8

1,768

57 x 89

5x8 = 40
7x9 = 63
5x9+7x8 = 101

40 101 63

5,073

15 x 932

9,320 + 9,320 / 2
9,320 + 4,500 + 150 + 10
9,320 + 4,660

13,980

31 x 42

3x4 = 12
1x2 = 2
3x2 + 1x4 = 10

12 10 2

1,302

BALANCING & ROUNDING AND COMPENSATION

71 x 27

(71x3) x (27/3)

217x9

1,917

81 x 882

(81/9) x (882x9)

9 x 7,938

71,442

62 x 740

(60 x 740) + (2 x 740)

44,400 + 1,480

45,880

344 x 44

(344x4) x (44/4)

1,376 x 11

15,136

121 x 5159

(121 / 11) x (5,159 x 11)

11 x 56,749

624,239

101 x 341

(100 x 341) + (1x341)

34,100 + 341

34,441

25 x 70

(25/5) x (70x5)

5 x 350

1,750

8 x 583

(8 x 600) - (17x8)

4,800 - 136

4,664

97 x 432

(100 x 432) - (3x432)

43,200 - 1,296

41,904

444 x 36

(444x6) x (36/6)

2,664x6

15,984

31 x 53

(30 x 53) - (1x53)

1,643

9 x 999

(10 x 999) - (1x999)

8,991

ASSOCIATING AND DISTRIBUTING

43 x 444

(43 x 4) x 111
172 x 111

19,092

663 x 58

6630 x 6 - 2 x 663
39780 - 1326

38,454

861 x 34

861 x 30 + 4 x 861
25,830 + 3,444

29,274

312 x 45

104 x (3 x 45)
104 x 135
100 x 135 + 4 x 135
13,500 + 540

14,040

35 x 858

8,58 x 30 + 5 x 858
25,740 + 4,290

30,030

77 x 96

77 x 100 - 7 x 96
7,700 - 308

7,392

382 x 820

382 x 10 x 82
3,820 x 82
3,820 x 80 x 2 x 3820
305,600 + 7,640

313,240

53 x 25

50 x 25+x3 x 25
1250 + 75

1,325

950 x 81

950 x 80 + 1x950
76,000 + 950

76,950

84 x 583

5,83 x 80 + 4 x 583
46,640 + 2,332

48,972

712 x 63

7,120 x 6 + 3 x 712
42,720 + 2,136

44,856

997 x 24

24 x 1,000 – 3 x 24
24,000 - 3 x 24
24,000 – 72

23,928

49 x 17

170 x 5 - 1 x 17
850 - 17

833

TRIPLE DIGIT PRODUCTS

212 x 485

2x5 = 10
1x5 + 2x8 = 21
2x5 + 1x8 + 2x4 = 26
2x8 + 1x4 = 20
1x4 = 8

8 20 26 21 10

102,820

230 x 484

Try mentally computing

8 28 32 12 0

111,320

485 x 950

36 92 85 25 0

460,750

667 x 193

7x3 = 21
6x3 + 7x9 = 81
6x3 + 6x9 + 7x1 = 79
6x9 + 6x1 = 60
6x1 = 6

6 60 79 81 21

128,731

431 x 345

12 25 35 19 5

148,695

PRODUCTS WITH MULTIPLICANDS CLOSE TO POWERS OF 10

375 x 99

37,500 - 375

37,125

or

375 - 1 = 374
100 - 375 = -275

37,400
-275

37,125

891 x 9,999

8,910,000 - 891

8,909,109

or

891 - 1 = 890
10,000 - 891 = 9,109

8,909,109

998 x 999

Try it mentally

-2, -1

99 + (-1) = 997
(-2) x (-1) = 2

997,002

462 x 999

462,000 - 462

461,538

or

462 - 1 = 461
1,000 - 462 = 538

461,538

97 x 98

97 - 100 = -3,
98 - 100 = -2,
97 + (-2) = 95
(-3) x (-2) = 6

9,506

9,998 x 9,997

-2, -3
9,998 + (-3) = 9,995
(-2) x (-3) = 6

99,950,006

APPENDIX A2: WORKED OUT ANSWERS TO PROBLEMS (CHAPTER 6)

SINGLE DIGIT DIVISORS

473 / 4
236.5
118.25

474 / 3
158
(15 x 3 fits into 47)
(8x3 fits into 24)
158

316 / 4
79
(7x4 fits into 31)
(9x4 fits into 36)
79

384 / 6
64
(64 x 6 fits into 384)
64

479 / 7
68.428571
(6x7 fits into 47)
(8x7 fits into 59)
(3/7 is 0.428571)
68.428571

581 / 8
290.5
145.25
72.625

219 / 9
24.33
(2x9 fits into 24)
(4x9 fits into 39)
(3x9 fits into 30)
(3x9 fits into 30)
24.33

444 / 8
55.5
(5x8 fits into 44)
(5x8 fits into 44)
(5x8 fits into 40)
55.5

999 / 6
166.5
(16 x 6 fits into 99)
(6x6 fits into 39)
(5x6 fits into 30)
166.5

123 / 7
17.571428
(1x7 fits into 12)
(7x7 fits into 53)
(4/7 is 0.571428)
17.571428

4,560 / 8
2,280
1,140
570

1,311 / 3
437
(43x3 fits into 131)
(7x3 fits into 21)
437

3,808 / 4
952
(95x4 fits into 380)
(2x4 fits into 8)
952

747 / 6
124.5
(12 x 6 fits in 74)
(4x6 fits into 27)
(5x6 fits into 30)
124.5

282 / 7
40.285714
(40 X 7 fits into 282)
2/7 is 0.285714
40.285714

5,738 / 4
2,869
1,434.5

3,819 / 9
424.33
(42x9 fits into 318)
(4x9 fits into 39)
(3x9 fits into 30)
(3x9 firs into 30)
424.33

4,796 / 8
599.5
(5x8 fits into 47)
(9x8 fits into 79)
(9x8 fits into 76)
(8 x 0.5 fits into 4)
599.5

RULE OF 11 AND ROUNDING AND COMPENSATION

242 / 11
22

489 / 4
122.25
500 x ½ is 250 x ½ is 125
11 x ½ is 5.5 x ½ is 2.75
125 - 2.75
122.25

1,421 / 7
203
1,400 / 7 is 200
21 / 7 is 3
200 + 3
203

847 / 11
77

1,157 / 6
192.83
1,200 / 6 is 200
43 / 6 is 7.17
200 0 7.17
192.83

5,589 / 5
1,117.8
5,500 / 5 is 1100
89 / 5 is 17.8
1,100 + 17.8
1,117.8

1,089 / 11
99

2,678 / 3
892.67
2,700 / 3 is 900
22 / 3 is 7.33
900 - 7.33
892.67

ASSOCIATING AND DISTRIBUTING IN DIVISION

105 / 15
105 / 5/3
21 / 3
7

1,584 / 16
1,584 / 2/2/2/2
792 / 2/2/2
396 / 2/2
198 / 2
99

749 / 7
700 / 7 + 49 / 7
100 + 7
107

1,015 / 5
1,000 / 5 + 15 / 5
200 + 3
203

14,271 / 71
14,200 / 71 + 71 / 71
200 + 1
201

516 / 12
516 / 2/2/3
258 / 2/3
129 / 3
43

4,956 / 21
4,956 / 3/7
1,652 / 7
235

621 / 3
600 / 3 + 21 / 3
200 + 7
207

4,998 /49
4,900 / 49 + 98 / 49
100 + 2
102

9,797 / 97
9,700 / 97 + 97 / 97
100 + 1
101

2,130 / 30
2,130 / 10/3
213 / 3
71

49,728 / 64
49,728 / 8/8
(6x8 fits into 49)
(2x8 fits into 17)
(1x8 fits into 12)
(6x8 fits into 48)
6216 / 8
(7x8 fits into 62)
(7x8 fits into 61)
(7x8 fits into 56)
777

BALANCING

117 / 39
39 is prime
117 / 39
3

1,616 / 404
4

70 / 14
70 / 2 / 14 / 2
45 / 7
5

171 / 57
57 is prime
171 / 57
3

588 / 294
588 / 3 / 294 / 3
196 / 98
196 /2 / 98 / 2
98 / 49
2

420 / 64
420 / 4 / 64 / 4
105 / 16
6 .56

LARGE DIVIDENDS

4,635 / 3
1,545
(1x3 fits into 4)
(5x3 fits into 16)
(4x3 fits into 13)
(5x3 fits into 15)
1545

2,928 / 4
732
(7x4 fits into 29)
(3x4 fits into 12)
(2x4 fits into 8)
732

6,534 / 66
99
(66 into 653 is 9, %59)
(66 into 594 is 9, %0)
99

4,995 / 5
999
(9x5 fits into 49)
(9x5 fits into 49)
(9x5 fits into 45)
999

21,792 / 93
234.32
(93 into 217 is 2, %31)
(93 into 319 is 3, %40)
(93 into 402 is 4, %30)
(93 into 300 is 3, %21)
(93 into 210 is 2, %24)
24 is less than half of 93
so no rounding up
234.32

5,929 / 77
77
(77 into 592 is 7, %53)
(77 into 539 is 7, %0)
77

522/ 6
87
(8x6 fits into 52)
(7x6 fits into 42)
87

4,992 / 48
104
(48 into 49 is 1, %1)
(48 into 19 is 0)
(48 into 192 is 4, %0)
104

4,575 / 61
75
(61 into 457 is 7, %30)
(61 into 305 is 5, %0)
75

40,131 / 63
637
(63 into 401 is 6, %23)
(63 into 233 is 3, %44)
(63 into 441 is 7, %0)
637

10,248 / 21
488
(21 into 102 is 4, %18)
(21 into 184 is 8, %16)
(21 into 168 is 8, %0)
488

48,179 / 18
26,76.61
(18 into 48 is 2, %12)
(18 into 121 is 6, %13)
(18 into 137 is 7, %11)
(18 into 119 is 6, %11)
(18 into 110 is 6, %2)
(18 into 20 is 1, %2)
the two last reminders are the
same, we can predict
a decimal periodic number
2676.61

48840 / 88
555
(88 into 488 is 5, %48)
(88 into 484 is 5, %44)
(88 into 440 is 5, %0)
555

12,845 / 55
233.55
(55 into 128 is 2, %18)
(55 into 184 is 3, %19)
(55 into 195 is 3, %30)
(55 into 300 is 5, %25)
(55 into 250 is 4, %30)
30 is more than half 55
so round up
233.55

10,384 / 32
324.5
(32 into 103 is 3, %7)
(32 into 78 is 2, %14)
(32 into 144 is 4, %16)
(32 into 160 is 5, %0)
324.5

DIVISORS LARGER THAN DIVIDENDS

1 / 99
100 / 99
0.0…
(99 into 100 is 1, %1)
(99 into 10 is 0)
(99 into 100 is 1, %1)
we can tell 01 is periodic
0.0101

7 / 32
70 / 32
0. …
(32 into 70 is 2, %6)
(32 into 60 is 1, %28)
(32 into 280 is 8, %24)
(32 into 240 is 7, %16)
16 is exactly half of 32
so round up
0.2188

5 / 783
5,000 / 783
0.00 …
(783 into 5000 is 6, %302)
(783 into 3020 is 3, %651)
651 is more than half of 783,
so round up
0.0064

3 / 47
300 / 47
0.0…
(47 into 300 is 6, %18)
(47 into 180 is 3, %39)
(47 into 390 is 8, %14)
14 is less than half of 47
so don't round up
0.0638

6 / 49
60 / 49
0. …
(49 into 60 is 1, %11)
(49 into 110 is 2, %12)
(49 into 120 Is 2, %22)
(49 into 220 is 4, %24)
24 is less than half of 49
so don't round up
0.1224

9 / 2,903
9,000 / 2,903
0.00 …
(2,903 into 9,000 is 3, %291)
(2,903 into 2,910 is 1, %7)
7 is less than half of 2,903
so don't round up
0.0031

5 / 36
50 / 36
0. …
(36 into 50 is 1, %14)
(36 into 140 is 3, %32)
(36 into 320 is 8, %32)
(36 into 320 is 8, %32)
we can tell that 8 is periodic,
we round up.
0.1389

3 / 481
3000 / 481
0.00 …
(481 into 3000 is 6, %114)
(481 into 1140 is 2, %154)
154 is less than half 481
so don't round up
0.0062

10 / 3,841
10,000 / 3,841
0.00
3,841 into 10,000 is 2,
%2,318
3,841 into 23,180 is 6, %134
134 is less than half 38,41
so don´t round up
0.0026

15 / 1,374
1,500 / 1,374
0.0 …
(1374 into 1,500 is 1, %126)
(1374 into 1,260 is 0)
(1374 into 12,600 is 9, %234)
234 is less than half of 1,374
so don't round up
0.0109

594 / 671
5940 / 671
0. …
(671 into 5940 is 8, 572)
(671 into 5720 is 8, 352)
(671 into 3520 is 5, 165)
(671 into 1650 is 2, 308)
308 is less than half 671
so don't round up
0.8852

23 / 2,323
23,000 / 2,323
0.00 …
(2,323 into 23,000 is 9, %2093)
(2,323 into 20,930 is 9, %23)
(2,323 into 230 is 0)
we can infer the sequence
0099 will be periodic
0.0099

19 / 40,100
190,000 / 40,100
0.000 …
(40,100 into 190,000 is 4, %29600)
(29,600 is more than half of 40,100 so round up)
0.0005

11 / 1111
11,000 / 1,111
0.00 …
(1,111 into 11,000 is 9, %1001)
(1,111 into 10,011 is 9, %12)
12 is less than half of 1,111
so don't round up
0.0099

10 / 9,999
10,000 / 9,999
0.00
(9,999 into 10,000 is 1, %1)
(9,999 into 10 is 0)
(9,999 into 100 is 0)
we can predict the sequence
001 will be periodic
0.0010

203 / 425
2,030 / 425
0. …
(425 into 2,030 is 4, %330)
(425 into 3,300 is 7, %325)
(425 into 3,250 is 7, %275)
(425 into 2,750 is 6, %200)
200 is less than half of 425
so don´t round up
0.4776

99 / 3,847
9,900 / 3,847
0.0
(3,847 into 9,900 is 2, %2,206)
(3,847 into 22,060 is 5, %825)
(3,847 into 28,250 is 7, %1321)
1,321 is less than half of 3,847
0.0257

4,839 / 14,748
48,390 / 14,748
0. …
(14,748 into 48,390 is 3, %4,146)
(14,748 into 41,460 is 2, %11,964)
(14,748 into 119,640 is 8, %1656)
(14,748 into 16,560 is 1, %1,812)
1,812 is less than half of 14,748, don't round up
0.3281

2,244 / 3,399
22,440 / 3,399
0. …
(3,399 into 22,440 is 6, %2046)
(3,399 into 20,460 is 6, %66)
(3,399 into 660 is 0)
(3,399 into 6,600 is 1, %3,201)
3,201 is more than half 3,391, so round up
0.6602

11 / 121
1,100 / 121
0.0 …
(121 into 1100 is 9, %11)
(121 into 110 is 0)
(121 into 1100 is 9, %11)
we can predict the sequence 09 will be periodic
0.0909

9,999 / 99,999
999,900 / 99,999
0.0 …
(99,999 into 999,900 is 9, %99,909)
(99,999 into 999,090 is 9, %99,099)
(99,999 into 990,990 is 9, %90,999)
90,999 is more than half of 99,999, so round up
(*0.09999* to **0.1000)**

5,837 / 9,891
58,370 / 9,891
0. …
(9,891 into 58,370 is 5, %8,915)
(9,891 into 89,150 is 9, %131)
(9,891 into 1,310 is 0)
(9,891 into 13,100 is 1, % 3,209)
3,209 is less than half 9,891
so don't round up
0.5901

17 / 34,747
170,000 / 34,747
0.000 …
(34,747 into 170,000 is 4, %31,012)
31,012 is more than half 34,747 so round up
0.0005

999 / 1,000
9,990 / 1,000
0. …
(1,000 into 9,990 is 9, %990)
(1,000 into 9,900 is 9, %900)
(1,000 into 9,000 is 9, %0)
0.9990

5,783 / 5,863
57,830 / 5,863
0. …
(5,863 into 57,830 is 9, %5,063)
(5,863 into 50,630 is 8, %3726)
(5,863 into 37,260 is 6, %2,082)
(5,863 into 20,820 is 3, %3,231)
3,231 is more than half of 5,863
so round up
0.9864

APPENDIX B: ALGEBRAIC PROPERTIES OF PRODUCTS AND QUOTIENTS

ALGEBRAIC PROPERTIES OF MULTIPLICATION AND DIVISION	
COMMUTATIVE PROPERTY	
a x b = b x a	2 x 3 = 3 x 2
ASSOCIATIVE PROPERTY	
(a x b) x c = a x (b x c)	(2 x 3) x 5 = 2 x (3 x 5) 6 x 5 = 2 x 15
DISTRIBUTIVE PROPERTY	
a x (b + c) = a x b + a x c	2 x (3 + 4) = 2 x 3 + 2 x 4 2 x 7 = 6 + 8
IDENTITY PROPERTY	
a x 1 = a	2 x 1 = 2
INVERSE OPERATIONS	
a x 1/a = 1	2 x 1/2 = 1
PROPERTIES OF ZERO	
a x 0 = 0	2 x 0 = 0
a / 0 = undefined	2 / 0 = undefined

APPENDIX C1: BASIC MULTIPLICATION TABLES

BASIC MULTIPLICATION TABLES

x	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

APPENDIX C2: BASIC DIVISION TABLES

BASIC DIVISION TABLES – TWO DIGITS I

79	78	77	76	75	74	73	72	71	70	69	68	67	66	65	64	63	62	61	60	/
7.2	7.1	7	6.9	6.8	6.7	6.6	6.5	6.5	6.4	6.3	6.2	6.1	6	5.9	5.8	5.7	5.6	5.5	5.5	11
6.6	6.5	6.4	6.3	6.3	6.2	6.1	6	5.9	5.8	5.8	5.7	5.6	5.5	5.4	5.3	5.3	5.2	5.1	5	12
6.1	6	5.9	5.8	5.8	5.7	5.6	5.5	5.5	5.4	5.3	5.2	5.2	5.1	5	4.9	4.8	4.8	4.7	4.6	13
5.6	5.6	5.5	5.4	5.4	5.3	5.2	5.1	5.1	5	4.9	4.9	4.8	4.7	4.6	4.6	4.5	4.4	4.4	4.3	14
5.3	5.2	5.1	5.1	5	4.9	4.9	4.8	4.7	4.7	4.6	4.5	4.5	4.4	4.3	4.3	4.2	4.1	4.1	4	15
4.9	4.9	4.8	4.8	4.7	4.6	4.6	4.5	4.4	4.4	4.3	4.3	4.2	4.1	4.1	4	3.9	3.9	3.8	3.8	16
4.6	4.6	4.5	4.5	4.4	4.4	4.3	4.2	4.2	4.1	4.1	4	3.9	3.9	3.8	3.8	3.7	3.6	3.6	3.5	17
4.4	4.3	4.3	4.2	4.2	4.1	4.1	4	3.9	3.9	3.8	3.8	3.7	3.7	3.6	3.6	3.5	3.4	3.4	3.3	18
4.2	4.1	4.1	4	3.9	3.9	3.8	3.8	3.7	3.7	3.6	3.6	3.5	3.5	3.4	3.4	3.3	3.3	3.2	3.2	19
4	3.9	3.9	3.8	3.8	3.7	3.7	3.6	3.6	3.5	3.5	3.4	3.4	3.3	3.3	3.2	3.2	3.1	3.1	3	20

99	98	97	96	95	94	93	92	91	90	89	88	87	86	85	84	83	82	81	80	/
9	8.9	8.8	8.7	8.6	8.5	8.5	8.4	8.3	8.2	8.1	8	7.9	7.8	7.7	7.6	7.5	7.5	7.4	7.3	11
8.3	8.2	8.1	8	7.9	7.8	7.8	7.7	7.6	7.5	7.4	7.3	7.3	7.2	7.1	7	6.9	6.8	6.8	6.7	12
7.6	7.5	7.5	7.4	7.3	7.2	7.2	7.1	7	6.9	6.8	6.8	6.7	6.6	6.5	6.5	6.4	6.3	6.2	6.2	13
7.1	7	6.9	6.9	6.8	6.7	6.6	6.6	6.5	6.4	6.4	6.3	6.2	6.1	6.1	6	5.9	5.9	5.8	5.7	14
6.6	6.5	6.5	6.4	6.3	6.3	6.2	6.1	6.1	6	5.9	5.9	5.8	5.7	5.7	5.6	5.5	5.5	5.4	5.3	15
6.2	6.1	6.1	6	5.9	5.9	5.8	5.8	5.7	5.6	5.6	5.5	5.4	5.4	5.3	5.3	5.2	5.1	5.1	5	16
5.8	5.8	5.7	5.6	5.6	5.5	5.5	5.4	5.4	5.3	5.2	5.2	5.1	5.1	5	4.9	4.9	4.8	4.8	4.7	17
5.5	5.4	5.4	5.3	5.3	5.2	5.2	5.1	5.1	5	4.9	4.9	4.8	4.8	4.7	4.7	4.6	4.6	4.5	4.4	18
5.2	5.2	5.1	5.1	5	4.9	4.9	4.8	4.8	4.7	4.7	4.6	4.6	4.5	4.5	4.4	4.4	4.3	4.3	4.2	19
5	4.9	4.9	4.8	4.8	4.7	4.7	4.6	4.6	4.5	4.5	4.4	4.4	4.3	4.3	4.2	4.2	4.1	4.1	4	20

BASIC DIVISION TABLES – TWO DIGITS II

73	72	71	70	69	68	67	66	65	64	63	62	61	60	/
73	72	71	70	69	68	67	66	65	64	63	62	61	60	1
36.5	36	35.5	35	34.5	34	33.5	33	32.5	32	31.5	31	30.5	30	2
24.3	24	23.7	23.3	23	22.7	22.3	22	21.7	21.3	21	20.7	20.3	20	3
18.3	18	17.8	17.5	17.3	17	16.8	16.5	16.3	16	15.8	15.5	15.3	15	4
14.6	14.4	14.2	14	13.8	13.6	13.4	13.2	13	12.8	12.6	12.4	12.2	12	5
12.2	12	11.8	11.7	11.5	11.3	11.2	11	10.8	10.7	10.5	10.3	10.2	10	6
10.4	10.3	10.1	10	9.86	9.71	9.57	9.43	9.29	9.14	9	8.86	8.71	8.57	7
9.13	9	8.88	8.75	8.63	8.5	8.38	8.25	8.13	8	7.88	7.75	7.63	7.5	8
8.11	8	7.89	7.78	7.67	7.56	7.44	7.33	7.22	7.11	7	6.89	6.78	6.67	9
7.3	7.2	7.1	7	6.9	6.8	6.7	6.6	6.5	6.4	6.3	6.2	6.1	6	10

85	84	83	82	81	80	79	78	77	76	75	74	/
85	84	83	82	81	80	79	78	77	76	75	74	1
42.5	42	41.5	41	40.5	40	39.5	39	38.5	38	37.5	37	2
28.3	28	27.7	27.3	27	26.7	26.3	26	25.7	25.3	25	24.7	3
21.3	21	20.8	20.5	20.3	20	19.8	19.5	19.3	19	18.8	18.5	4
17	16.8	16.6	16.4	16.2	16	15.8	15.6	15.4	15.2	15	14.8	5
14.2	14	13.8	13.7	13.5	13.3	13.2	13	12.8	12.7	12.5	12.3	6
12.1	12	11.9	11.7	11.6	11.4	11.3	11.1	11	10.9	10.7	10.6	7
10.6	10.5	10.4	10.3	10.1	10	9.88	9.75	9.63	9.5	9.38	9.25	8
9.44	9.33	9.22	9.11	9	8.89	8.78	8.67	8.56	8.44	8.33	8.22	9
8.5	8.4	8.3	8.2	8.1	8	7.9	7.8	7.7	7.6	7.5	7.4	10

99	98	97	96	95	94	93	92	91	90	89	88	87	86	/
99	98	97	96	95	94	93	92	91	90	89	88	87	86	1
49.5	49	48.5	48	47.5	47	46.5	46	45.5	45	44.5	44	43.5	43	2
33	32.7	32.3	32	31.7	31.3	31	30.7	30.3	30	29.7	29.3	29	28.7	3
24.8	24.5	24.3	24	23.8	23.5	23.3	23	22.8	22.5	22.3	22	21.8	21.5	4
19.8	19.6	19.4	19.2	19	18.8	18.6	18.4	18.2	18	17.8	17.6	17.4	17.2	5
16.5	16.3	16.2	16	15.8	15.7	15.5	15.3	15.2	15	14.8	14.7	14.5	14.3	6
14.1	14	13.9	13.7	13.6	13.4	13.3	13.1	13	12.9	12.7	12.6	12.4	12.3	7
12.4	12.3	12.1	12	11.9	11.8	11.6	11.5	11.4	11.3	11.1	11	10.9	10.8	8
11	10.9	10.8	10.7	10.6	10.4	10.3	10.2	10.1	10	9.89	9.78	9.67	9.56	9
9.9	9.8	9.7	9.6	9.5	9.4	9.3	9.2	9.1	9	8.9	8.8	8.7	8.6	10